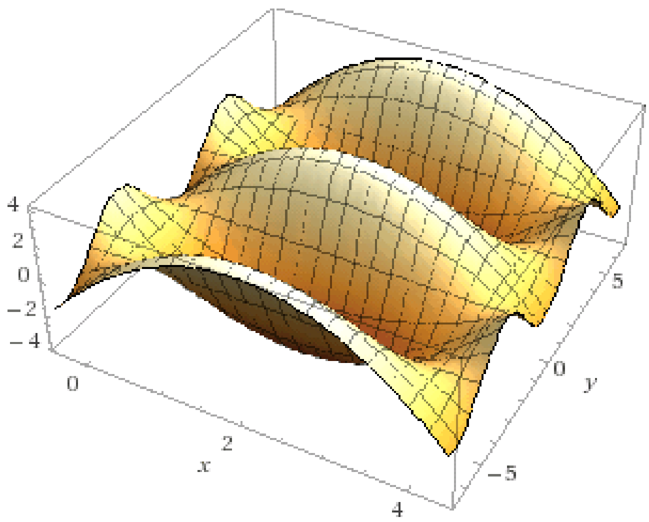
Problem set – Week 8

Extrema problems

1. Find the absolute extrema of the surface $f(x, y) = (4x - x^2)\cos(y)$ on the rectangular plate $1 \le x \le 3$, $-\pi/4 \le y \le \pi/4$.



2. A flat circular plate P of radius 1 is heated (included the boundary of the plate) so that the temperature at the point $(x, y) \in P$ is

$$T(x, y) \;=\; x^2 + 2y^2 - x.$$

Find the temperatures at the hottest and coldest points on the plate.

3. Find the numbers $a \le b$ such that the integral

$$\int_a^b \left(e^{x^2} - 2\right) dx.$$

has its largest value.

4. Find three numbers whose sum is 9 and whose sum of squares is a minimum.

5. Among all the points on the surface $z = 10 - x^2 - y^2$ that lie above the plane $x + 2y + 3z = 0$, find the point farthest from the plane.

6. The Hessian matrix of $f(x, y) = x^2y^2$ at $(0, 0)$ is the zero matrix. Determine whether the function has an extremum or not at the origin by imagining what the surface looks like.

7. In this exercise, we give a proof that the geometric mean is $\leq$ the arithmetic mean, for any set of n non-negative real numbers, i.e.

$$(a_1 \cdots a_n)^{1/n} \leq \frac{a_1 + \cdots + a_n}{n}. \tag{1}$$

(a) Explain why the maximum value of $x^2y^2z^2$ on a sphere of radius r centered at the origin is $(r^2/3)^3$.

(b) Deduce (1) from (a) for $n = 3$.

(c) Explain how the argument can be seen to hold more generally for any $n \geq 1$.