Solutions – Week 1

Linear systems & Gauss elimination

1. Solve the following linear systems via elimination.

$$(a)\ \left|\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 8 \\ x & + & 3y & + & 3z & = & 10 \\ x & + & 2y & + & 4z & = & 9 \end{array}\right| \qquad (b)\ \left|\begin{array}{ccccc} x & - & 2y & = & 2 \\ 3x & + & 5y & = & 17 \end{array}\right|$$

$$(c)\ \left|\begin{array}{ccccccc} x & + & 4y & + & z & = & 0 \\ 4x & + & 13y & + & 7z & = & 0 \\ 7x & + & 22y & + & 13z & = & 0 \end{array}\right| \qquad (d)\ \left|\begin{array}{ccccccc} x & + & 4y & + & z & = & 0 \\ 4x & + & 13y & + & 7z & = & 0 \\ 7x & + & 22y & + & 13z & = & 1 \end{array}\right|$$

Sketch the solutions of (b) graphically, as intersection of lines in the x-y-plane. Describe your solutions to (c) in terms of intersecting planes. Here are another two linear systems to solve.

$$(e)\ \left|\begin{array}{ccccc} x & + & y & = & 1 \\ 2x & - & y & = & 5 \\ 3x & + & 4y & = & 2 \end{array}\right| \qquad (f)\ \left|\begin{array}{ccccccccc} x_1 & + & & & 2x_3 & + & 4x_4 & = & -8 \\ & & x_2 & - & 3x_3 & - & x_4 & = & 6 \\ 3x_1 & + & 4x_2 & - & 6x_3 & + & 8x_4 & = & 0 \\ & - & x_2 & + & 3x_3 & + & 4x_4 & = & -12 \end{array}\right|$$

Solutions :

(a) Comparing first and third rows yields $z = 1$. Comparing first and second rows yields $y = 2$. Plugging $y = 2$ and $z = 1$ in the first row finally yields $x = 1$. The solution of this system is exactly the point $(1, 2, 1)$ in x-y-z-space.

(b) Subtracting three times the first row to the second, we obtain $11y = 11$. Plugging $y = 1$ in the first row, $x = 4$. The point $(4, 1)$ is exactly the intersection of the lines $\{y = \frac{x}{2} - 1\}$ and $\{y = -\frac{3}{5}x + \frac{17}{5}\}$.

(c) By elimination, we reduce the system to

$$\left|\begin{array}{ccccccc} x & + & 4y & + & z & = & 0 \\ & - & 3y & + & 3z & = & 0 \\ & - & 6y & + & 6z & = & 0 \end{array}\right|$$

and further to

$$\left|\begin{array}{ccccccc} x & + & 5y & & & = & 0 \\ & - & y & + & z & = & 0 \end{array}\right|$$

Each row describes a plane in x-y-z-space. The intersection of these two planes is a line, whose points are all solutions of the system. Set $y = t$ to be an arbitrary real number. Then $z = t$ and $x = -5t$. The line in question is then parametrized by $\{t(-5, 1, 1) : t \in \mathbb{R}\}$ and passes through the origin.

(d) The system reduces to

$$\left| \begin{array}{ccccccc} x & + & 4y & + & z & = & 0 \\ & - & 3y & + & 3z & = & 0 \\ & - & 6y & + & 6z & = & 1 \end{array} \right|$$

Comparing the last two rows, we can see that the system is inconsistent; it can have no solutions, because $1 = 0$ is never true.

(e) Summing up the first two rows, $3x = 6$. Plugging it in in any row, $y = -1$. Observe that there are more rows in this system than necessary to find its solution.

(f) We reduce the system to

$$\left| \begin{array}{ccccc} x_1 & + & 2x_3 & = & 0 \\ x_2 & - & 3x_3 & = & 4 \\ & & x_4 & = & -2 \end{array} \right|$$

with solution the line $(0, 4, 0, -2) + t(-1, 3, 1, 0)$.

2. Consider the linear system

$$\left| \begin{array}{ccccccc} x & + & y & - & z & = & -2 \\ 3x & - & 5y & + & 13z & = & 18 \\ x & - & 2y & + & 5z & = & k \end{array} \right|$$

where k is an arbitrary constant.

(a) For which value(s) of k does this system have one or infinitely many solutions ?

(b) For each of these values, how many solutions does the system have ?

(c) Write down all solutions.

Solutions :

(a) The first reductions yield

$$\left| \begin{array}{ccccccc} x & + & y & - & z & = & -2 \\ & & y & - & 2z & = & -3 \\ & & & & 0 & = & k-7 \end{array} \right|$$

hence $k = 7$ is a necessary condition for the system to be consistent.

(b) With $k = 7$, the last row now amounts to $0 = 0$ and we can discard it. We are left with two equations of three variables, that is, geometrically, two planes. If these two equations admit a simultaneous solution, the two planes intersect in a line, of which each point is a solution. There are an infinity of solutions.

(c) The line is parametrized as $(1, -3, 0) + t(-1, 2, 1)$, $t \in \mathbb{R}$.

3. Why are linear systems particularly easy to solve when they are in triangular form ? Answer by considering the upper triangular system

$$\left| \begin{array}{ccccccccc} x_1 & + & 2x_2 & - & x_3 & + & 4x_4 & = & -3 \\ & & x_2 & + & 3x_3 & + & 7x_4 & = & 5 \\ & & & & x_3 & + & 2x_4 & = & 2 \\ & & & & & & x_4 & = & 0 \end{array} \right|$$

Solution : Proceed by backwards substitution : The last row tells you that $x_4 = 0$. Plug in this solution in the second-to-last row to obtain the value of x_3. Do similarly in the second row by plugging in the values you have for x_3 and x_4, you now know x_2, and so forth.

4. Find a system of linear equations with three unknowns whose solutions are the points on the line through $(1, 1, 1)$ and $(3, 5, 0)$.

Solution : The line passing through points $(1, 1, 1)$ and $(3, 5, 0)$ is parametrized by

$$L \;:\; \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha \left(\begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}, \qquad \alpha \in \mathbb{R}.$$

We wish to set up a system of linear equations in three unknowns x, y, z, of which $x = 1 + 2\alpha$, $y = 1 + 4\alpha$, $z = 1 - \alpha$ are solutions. Observe that this amounts to

$$x = -2z + 3, \qquad y = 2x - 1.$$

Then for the linear system described by the two planes

$$\left| \begin{array}{ccccc} x & + & 2z & = & 3 \\ 2x & - & y & = & 1 \end{array} \right|$$

each point on the line L is a solution.

5. We call a function f a polynomial of degree 2 if it is of the form $f(t) = at^2 + bt + c$, with $a \neq 0$. Find the polynomial of degree 2 whose graph passes through the points $(-1, 1)$, $(2, 3)$ and $(3, 13)$ in the x-y-plane.

Solution : We want to find a polynomial $f(t) = at^2 + bt + c$ such that $f(-1) = 1$, $f(2) = 3$, $f(3) = 13$. This amounts to solve the linear system

$$\left| \begin{array}{ccccccc} a & - & b & + & c & = & 1 \\ 4a & + & 2b & + & c & = & 3 \\ 9a & + & 3b & + & c & = & 13 \end{array} \right|$$

The resulting polynomial is $f(t) = \frac{7}{3}t^2 - \frac{5}{3}t - 3$.

6. Let us assume that parking meters in Zurich only accept coins of 20ct, 50ct and 1 Fr. As an incentive, the city council offers a reward to any patrolman who, from his daily round, brings back exactly 1000 coins, worth exactly 1000 Fr.

   What are the odds for this reward to be claimed any time soon ?

   Solution : Let x denote the number of 20ct coins, y the number of 50ct coins and z the number of 1fr. coins. The two conditions to claim the reward are

$$\left| \begin{array}{ccccccc} x & + & y & + & z & = & 1000 \\ \frac{1}{5}x & + & \frac{1}{2}y & + & z & = & 1000 \end{array} \right|$$

   The solution of the system is $x = -\frac{5}{3}(1000 - z)$, $y = \frac{8}{3}(1000 - z)$. Note that x, by definition, needs to be a non-negative integer and that since $z \leqslant 1000$, $x \geqslant 0$ forces $z = 1000$. The only possible solution is $x = 0$, $y = 0$, $z = 1000$.