Solutions – Week 12

First-order differential equations

1. Solve the following differential equations.

   (a) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$
   Solution : $y(x) = \frac{x+c}{1-cx}$, where c is a constant.

   (b) $\frac{dy}{dx} = \frac{\cos^2 y}{\sin^2 x}$
   Solution : $y(x) = \arctan\left(c - \frac{1}{\tan(x)}\right)$

2. Describe geometrically the set of curves that are orthogonal to the integral curves for the differential equation $ydx = xdy$.

   Solution : All concentric circles centered at the origin.

3. Write down a differential equation of the form $y' = f(y)$ with solution

   (a) $y(x) = x^\alpha$
   Solution : $y'(x) = \frac{\alpha}{x} y(x)$

   (b) $y(x) = \ln(x)$
   Solution : $y' = \frac{1}{e^y}$

   (c) $y(x) = \tan(x)$
   Solution : $y' = 1 + y^2$

   (d) $y(x) = \arcsin(x)$
   Solution : $y' = (1 - \sin^2(y))^{-1/2}$

4. Solve the differential equation

$$\frac{dy}{dx} = \frac{2 - \sin(x+2y)}{2\sin(x+2y)}.$$

   Solution : $y(x) = \frac{1}{2}\left(\arccos(c - 2x) - x\right)$.

5. Let α be a real number and consider the initial value problem (IVP)

$$(*) \qquad \frac{dy}{dx} = y^\alpha, \quad y(0) = 0.$$

   (a) Show that this IVP has no solution if $\alpha = 1$.
   Solution : The solution is $y = e^c e^x$ for some constant c. The initial condition is never satisfied.

   (b) For $\alpha \neq 1$, determine the integral curve for $(*)$.
   Solution : $y = ((1-\alpha)x)^{\frac{1}{1-\alpha}}$

(c) Find the condition on α for $(*)$ to have a solution $y(x)$ defined for all $x \geq 0$.

Solution : For the solution above to exist, it is necessary that $(1-\alpha)x \geq 0$. Hence, we need $\alpha < 1$.

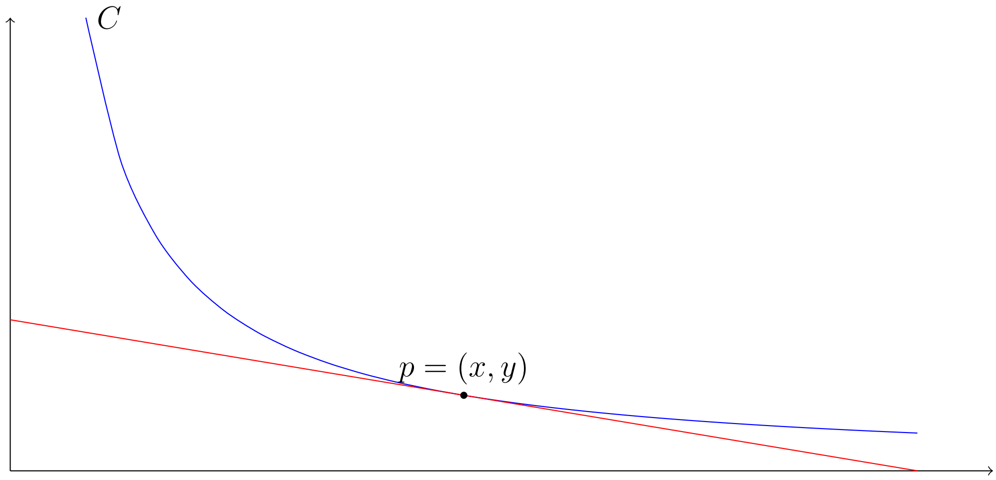
(d) Give an α for which $(*)$ has two solutions.

Solution : Take $\alpha = -1$. Then $y = \pm\sqrt{2x}$. More generally, for any α of the form $\alpha = -(2k-1)$, $k = 1, 2, \ldots$, there will be two solutions.

6. Find a curve C passing through the point $(3, 2)$ with the property that each point p on C is exactly the midpoint of the tangent line to C at p in the first quadrant.

(a) Sketch what C should look like.

Solution :



(b) Let $p = (x, y)$ be a point on C. Determine the slope of the tangent line to C at p.

Solution : The slope is $-y/x$.

(c) Set up the initial value problem for which C is an integral curve.

Solution : $y' = -\frac{y}{x}$ with $y(3) = 2$.

(d) Determine the equation of the curve by solving the IVP from (c).

Solution : $y(x) = 6/x$ is the unique solution on the domain $(0, \infty)$.

7. An executive conference room of a corporation contains 125 m^3 of air initially free of carbon monoxide (CO). Starting at time $t = 0$, cigarette smoke containing 4% CO is blown into the room at a rate of $r = .005$ m^3/min. A ceiling fan keeps the air in the room circulating so that it leaves the room at the same rate r. How long does it take for the concentration of CO in the room to reach .01% ?

Solution : Let $y(t)$ model the amount of carbon monoxide in that conference room at time t. The rate of change of y, $\frac{dy}{dt}$, is given by the difference of

- the incoming CO being added to the air by smoking, exactly $\frac{4}{100}(.005)$ m^3/min

- the amount of CO present in the air leaving he room through aeration ; the concentration is $\frac{y}{125}$ and the rate of aeration is again .005, hence

$$y' = \frac{4}{100}(.005) - \frac{.005}{125}y$$

is the rate of change described in the problem.

This is a differential equation of the form $y' + Ay = B$ for some specific constants A and B. The initial condition is $y(0) = 0$. One computes that

$$y(t) \;=\; \frac{B}{A}\left(1 - e^{-At}\right)$$

for all $t \geq 0$.

It will take 62 minutes and 58 seconds for the concentration of carbon monoxide in the room to reach .01 %.