Solutions – Week 2

Linear dependence & matrix multiplication

1. A bit of vector algebra.

   (a) Find all vectors perpendicular to $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$. Drawing a sketch might help !

   Solution : All vectors in the plane described by the equation $x+3y-z=0$.

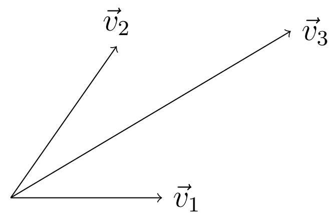
   (b) Find all solutions x_1, x_2, x_3 of the equation $\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$, where

$$\vec{b} = \begin{pmatrix} -8 \\ -1 \\ 2 \\ 15 \end{pmatrix}, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 4 \\ 7 \\ 5 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 8 \\ 3 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 4 \\ 6 \\ 9 \\ 1 \end{pmatrix}.$$

   Solution : The linear system has unique solution $x_1 = 12$, $x_2 = 3$, $x_3 = -4$.

   (c) Are $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ linearly dependent ?
   Solution : No. The equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = 0$ has unique solution 0.

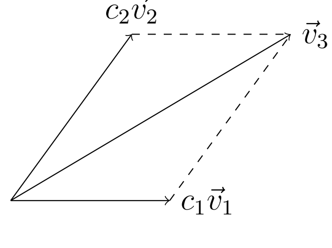
2. Consider the three vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ in the x-y-plane :



   Are $\vec{v}_1$, $\vec{v}_2$ linearly dependent ? What about $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$? Argue geometrically.

   Solution : The vectors $\vec{v}_1$, $\vec{v}_2$ are linearly dependent if and only if one is a scalar multiple of the other. But if $\vec{v}_2$ were a scalar multiple of $\vec{v}_1$, it would have to lie along the line going through $\vec{v}_1$. In the picture, this is clearly not the case, thus the two vectors are linearly independent.

   However, $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are linearly dependent, as with a correct scaling of $\vec{v}_1$ and $\vec{v}_2$, we get



3. Write the system

$$\left|\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 1 \\ 4x & + & 5y & + & 6z & = & 4 \\ 7x & + & 8y & + & 9z & = & 9 \end{array}\right|$$

in matrix form.

Solution :

$$\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}1\\ 4\\ 9\end{pmatrix}$$

4. If possible, compute the following matrix products.

(a) $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ (b) $\begin{pmatrix}1 & -1\\ -2 & 2\end{pmatrix}\begin{pmatrix}7 & 5\\ 3 & 1\end{pmatrix}$

(c) $\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ (d) $\begin{pmatrix}1 & -1\\ 0 & 2\\ 2 & 1\end{pmatrix}\begin{pmatrix}3 & 2\\ 1 & 0\end{pmatrix}$

(e) $\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ (f) $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$

(g) $\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 1\\ 1 & -1 & -2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\\ 2 & 1 & 3\end{pmatrix}$ (h) $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$

(i) $\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}-6 & 8\\ 3 & -4\end{pmatrix}$ (j) $\begin{pmatrix}1 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 1\\ 1 & 1\end{pmatrix}$

(k) $\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}3\\ 2\\ 1\end{pmatrix}$ (l) $\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\end{pmatrix}$

Solutions:

(a)	$\begin{pmatrix} 4 & 6 \\ 3 & 4 \end{pmatrix}$	(b)	$\begin{pmatrix} 4 & 4 \\ -8 & -8 \end{pmatrix}$
(c)	not possible	(d)	$\begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 7 & 4 \end{pmatrix}$
(e)	$\begin{pmatrix} a & b \\ c & d \\ 0 & 0 \end{pmatrix}$	(f)	$\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$
(g)	$\begin{pmatrix} -1 & 1 & 0 \\ 5 & 3 & 4 \\ -6 & -2 & -4 \end{pmatrix}$	(h)	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
(i)	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	(j)	$\begin{pmatrix} 0 & 1 \end{pmatrix}$
(k)	10	(l)	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$

5. Introducing inverses for 2×2 matrices.

   (a) Find all vectors $\vec{x}$ such that $A\vec{x} = \vec{b}$, where

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

   Solution : The matrix equation $A\vec{x} = \vec{b}$ corresponds to the linear system

$$\left| \begin{array}{ccccc} x & + & 2y & = & 2 \\ 3x & + & 4y & = & 1 \end{array} \right|$$

   which has the unique solution $\vec{x} = \begin{pmatrix} -3 \\ -5/2 \end{pmatrix}$.

   (b) Prove : The 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$.
   (**Hint :** Consider the cases $a = 0$ and $a \neq 0$ separately.)
   Solution : The matrix equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u & v \\ x & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

   yields the (simultaneous) linear systems (*)

$$\left| \begin{array}{ccccc} au & + & bx & = & 1 \\ cu & + & dx & = & 0 \end{array} \right| \qquad \left| \begin{array}{ccccc} av & + & by & = & 0 \\ cv & + & dy & = & 1 \end{array} \right|$$

   First assume that $a = 0$. Note that the system on the left is inconsistent if $b = 0$. Hence assume $b \neq 0$. This forces, in the system on the right, $y = 0$, in the system on the left, $x = 1/b$.

We are left with $cu = -d/b$ and $cv = 1$. Again, if $c = 0$, the whole system is inconsistent. Hence, assume $c \neq 0$ and the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ admits inverse $\begin{pmatrix} -d/bc & 1/c \\ 1/b & 0 \end{pmatrix}$ if and only if $bc \neq 0$.

Next, assume $a \neq 0$. Then one obtains from the linear system on the left (cf. (*)) $x(ad-bc) = -c$ and from the linear system on the right, $y(ad-bc) = a$. In particular, the total system is inconsistent if $ad - bc = 0$ and admits a unique solution otherwise.

(c) Prove : If A is invertible, then its inverse is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Solution : Observe that $ad - bc \neq 0$ is a necessary condition for the righthand side in the formula above to exist. By definition A is invertible if there exists a matrix B such that $AB = BA = 1$. You can check that the matrix A^{-1} described by the formula satisfies these two equations.

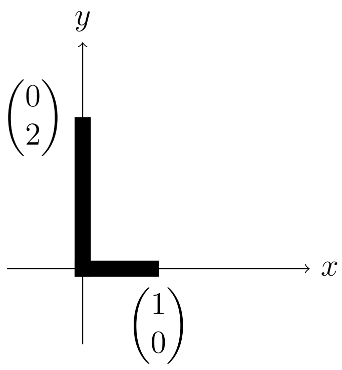
(d) Use the formula in (c) to compute $\vec{x}$ in (a).

Solution : Since $4 - 6 \neq 0$, the matrix A is invertible and $\vec{x} = A^{-1}\vec{b} = (-1/2)\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = -(1/2)\begin{pmatrix} 6 \\ -5 \end{pmatrix}$.

6. The goal of this exercise is to give a geometric interpretation of the linear transformations defined by the matrices

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Start by showing the effect of these transformations on the letter L :



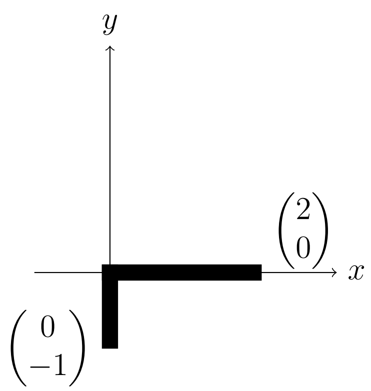
In each case, decide whether the transformation is invertible. Find the inverse if it exists, and interpret it geometrically.

Solutions : You can check that $A\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$, $B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$, $C\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$.

The matrix A scales vectors by a multiple of 3. You can see that the letter L, once we apply A to it, is still sitting at the origin but is now three times "bigger"; going on the vertical axis up to $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and on the horizontal axis to $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$. The inverse transformation would be a rescaling of $1/3$. In fact, using the formula from Exercise 5(c), you can check that the inverse matrix is given by $\begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix}$.

The matrix B projects points onto the horizontal axis. The letter L is reduced, under this projection, to the segment $[0, 1]$ on the x-axis. Intuitively, there should be no well-defined inverse, since any point on the vertical line passing by $(x, 0)$ is a potential pre-image of the transformation. Applying Exercise 5(b), you can in fact check that the criterion for inverses is not fulfilled by the matrix B.

To understand the geometric action of the matrix C, it may be easier to look at its effect on the letter L :



The letter L has been rotated by 90^o clockwise. The inverse is the rotation of 90^o counterclockwise, given by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.