Solutions – Week 3

Matrix multiplication, inverses & calculus warm-up

1. Let $A = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -4 & 7 \\ 3 & -5 \end{pmatrix}$.

   (a) Compute $5A - B$, A^2, A^{-1}, B^{-1}.

   Solutions : $\begin{pmatrix} 19 & -12 \\ -28 & 15 \end{pmatrix}$, $\begin{pmatrix} 14 & -5 \\ -25 & 9 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$, $\begin{pmatrix} 5 & 7 \\ 3 & 4 \end{pmatrix}$.

   (b) Show that $(AB)^{-1} = B^{-1}A^{-1}$.

   Solution : (To be) Discussed in class.

   (c) Check that $A^2 - 5A + I = 0$. Using this equation, find a formula for A^{-1} in terms of A.

   Solution : $A^{-1} = 5I - A$.

2. Find the matrix X such that $\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} X = \begin{pmatrix} 28 & 35 & 42 \\ 42 & 53 & 64 \end{pmatrix}$.

   Solution : $X = \begin{pmatrix} 0 & 1 & 2 \\ 7 & 8 & 9 \end{pmatrix}$.

3. Compute the first derivative of

$$\text{(a) } x^3 e^{-x^3} - x - 3, \qquad \text{(b) } \frac{\log(\sin^2(x))}{\cos(x)}, \qquad \text{(c) } \arctan(\sqrt{x}).$$

   Solutions :

$$\text{(a) } 3x^2 e^{-x^3}(1 - x^3) - 1, \quad \text{(b) } \frac{2}{\sin x} + \frac{\sin x}{\cos^2(x)} \log \sin^2(x), \quad \text{(c) } \frac{1}{1+x}.$$

   Compute the second derivative of

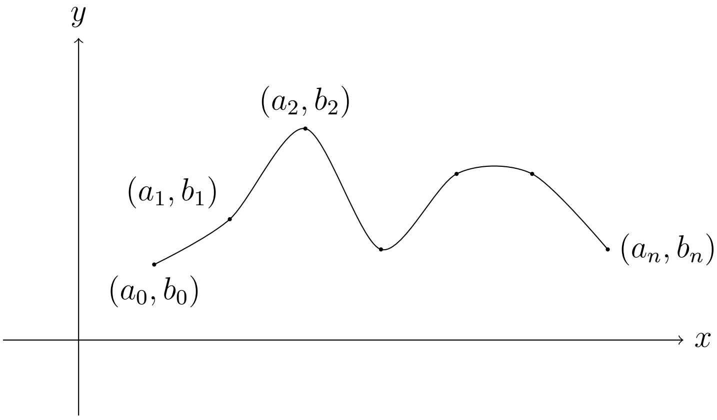
$$\text{(d) } \log(\log(x)).$$

   Solution : $\frac{-1}{x^2 \log(x)} \left(1 + \frac{1}{\log x}\right)$.

4. Find the equation of the line that is normal to the curve $y = (x^4 - 1)^3 \log(x + 1)$ at the origin.

   Solution : $y = x$.

5. You are in charge of designing a roller coaster ride. This simple ride is completely described, as viewed from the side, by the following figure.



You are given the set of points $(a_0, b_0), (a_1, b_1), \ldots, (a_n, b_n)$ and you need to connect these in a reasonably smooth way. A method often used is such design problems if that of cubic splines; you want to find polynomials $f_k(t)$ of degree at most 3, to define the shape of the ride between (a_{k-1}, b_{k-1}) and (a_k, b_k) by requiring $f_k(a_{k-1}) = b_{k-1}$ and $f_k(a_k) = b_k$ as well as $f_k'(a_k) = f_{k+1}'(a_k)$, $f_k''(a_k) = f_{k+1}''(a_k)$. Explain the practical significance of these conditions. For the convenience of the riders, it is also required that $f_1'(a_0) = f_n'(a_n) = 0$. Why ?

Show that satisfying all these conditions amounts to solving a linear system !

Solution : (To be) Discussed in class.

6. Sketch the curve $y = x^2(x-3)^2$.

   Solution : (To be) Discussed in class.

7. Let x, y be non-negative integers such that $x + y = 12$. What is the the maximal value of $x^2 y$?

   Solution : 256.

8. Compute the following integrals

$$\text{(a)} \int \tan(x)\ dx, \qquad \text{(b)} \int \frac{x\ dx}{x^4 + 2x^2 + 2}, \qquad \text{(c)} \int \sin^2(x)\ dx.$$

Solutions :

$$\text{(a)}\ -\log|\cos(x)| + c, \quad \text{(b)}\ \frac{1}{2}\arctan(x^2+1) + c, \quad \text{(c)}\ \frac{1}{2}x - \frac{1}{4}\sin(2x) + c.$$