

Problem set 1

Definition: For a map $f : X \rightarrow Y$, the *mapping cylinder* M_f is the quotient space of the disjoint union $Y \sqcup (X \times I)$ (where $I := [0, 1]$) obtained by identifying each $(x, 0) \in X \times I$ with $f(x) \in Y$.

The *mapping cone* C_f of f is defined as the quotient space $M_f / (X \times \{1\})$.

There are embeddings $i_X : X \rightarrow M_f; x \mapsto [x, 1]$ resp. $i_Y : Y \rightarrow M_f; y \mapsto [y]$ of X resp. Y into the mapping cylinder of f .

1. Let X be a contractible space and let $r : X \rightarrow A$ be a retraction. Show that A is also contractible.
2. Let $f, g : X \rightarrow S^n$ be maps such that $f(x) \neq -g(x)$ for all $x \in X$. Show that $f \simeq g$.
3. Let $f : X \rightarrow Y$ be a map. Suppose that there exist maps $g, h : Y \rightarrow X$ such that $f \circ g \simeq \text{id}_Y$ and $h \circ f \simeq \text{id}_X$. Show that f is a homotopy equivalence.
4. (a) Show that a map $\phi : M_f \rightarrow Z$ is continuous if and only if the induced maps $\phi_{X \times I} : X \times I \rightarrow Z$ and $\phi_Y : Y \rightarrow Z$ are continuous.
 (b) Let $f : X \rightarrow Y$ be a continuous map and let $r : M_f \rightarrow Y$ be the retract defined by $r([y]) = y$ and $r([x, t]) = f(x)$ for $y \in Y, x \in X$ and $t \in I$.

Show that there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{i_X} & M_f \\ & \searrow f & \downarrow r \\ & & Y \end{array}$$

such that $\text{id}_{M_f} \simeq i_Y \circ r$ and hence $M_f \simeq Y$.

- (c) Show that two spaces X and Y have the same homotopy type if and only if they can be embedded as (weak) deformation retracts of the same space.
5. Consider the map $f : S^1 \rightarrow S^1, e^{2\pi it} \mapsto e^{4\pi it}$ (or $z \mapsto z^2$). Prove that $C_f \simeq \mathbb{R}P^2$.