Prof. Paul Biran
ETH Zürich

Algebraic Topology I

HS 2015

Problem set 2

1. Let $A \subset X$ be a non-empty subset and assume that $\widetilde{H}_*(A) = 0$ (that is, A is acyclic). Prove that $H_*(X, A) \cong \widetilde{H}_*(X)$.

2. Suppose that X is a path-connected space and let $f : X \to X$ be a map. Prove that the induced map $f_* : H_0(X) \to H_0(X)$ is the identity.

3. Let $f : (X, x_0) \to (Y, y_0)$ be a map of pointed spaces and consider the induced maps $f_\sharp : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $f_* : H_1(X) \to H_1(Y)$. Prove commutativity of the diagram

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_\sharp} & \pi_1(Y, y_0) \\ \downarrow{\scriptstyle \phi_X} & & \downarrow{\scriptstyle \phi_Y} \\ H_1(X) & \xrightarrow{f_*} & H_1(Y) \end{array}$$

   where ϕ_X and ϕ_Y are the Hurewicz homomorphisms.

4. Let $p : X \to Y$ be a covering map, and let $x_0 \in X$ and $y_0 = p(x_0)$. Prove that the map $\pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a monomorphism. Is it true in general that $p_* : H_1(X) \to H_1(Y)$ is a monomorphism?

5. Let H be a "theory" satisfying axioms 1-4 of a homology theory, but not necessarily axiom 5. Show that

$$(i_X)_* \oplus (i_Y)_* : H_p(X) \oplus H_p(Y) \to H_p(X \sqcup Y)$$

   is an isomorphism for all spaces X, Y and for all $p \in \mathbb{Z}$, where $i_X : X \hookrightarrow X \sqcup Y$, $i_Y : Y \hookrightarrow X \sqcup Y$ denote the inclusions into the disjoint union.

   Hints.

   (a) Consider the long exact sequence of the pair $(X \sqcup Y, X)$.

   (b) Consider the excision $(Y, \emptyset) = \big((X \sqcup Y) \setminus X, X \setminus X\big) \xhookrightarrow{k} (X \sqcup Y, X)$ and the resulting isomorphism $k_* : H_*(Y) \xrightarrow{\cong} H_*(X \sqcup Y, X)$.

   (c) Note that the following diagram commutes:

$$\begin{array}{ccc} (Y, \emptyset) & \xrightarrow{\quad k \quad} & (X \sqcup Y, X) \\ & {\scriptstyle i_Y} \searrow \quad \nearrow {\scriptstyle j} & \\ & X \sqcup Y & \end{array}$$

(d) Deduce that in the long exact sequence

$$\cdots \to H_p(X) \xrightarrow{(i_X)_*} H_p(X \sqcup Y) \xrightarrow{j_*} H_p(X \sqcup Y, X) \to \dots$$

all maps j_* are surjective, and that thus the sequence gives rise to short exact sequences

$$0 \to H_p(X) \xrightarrow{(i_X)_*} H_p(X \sqcup Y) \xrightarrow{j_*} H_p(X \sqcup Y, X) \to 0.$$

(e) Find a right inverse for j_* to show that these short exact sequences split.