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Algebraic Topology I

HS 2015

Problem set 5

1. Let X be a space such that $H_i(X) = 0$ for all but finitely many $i \in \mathbb{Z}$. The *Euler characteristic* of X is defined as

$$\chi(X) := \sum_i (-1)^i \operatorname{rank} H_i(X).$$

   (a) Let X be a finite CW-complex and let a_i be the number of i-cells of X. Show that

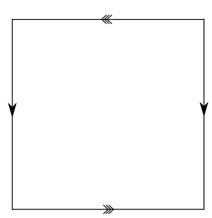
$$\chi(X) = \sum_i (-1)^i a_i.$$

   (b) Show that if $p : X \to Y$ is a k-sheeted covering map and Y is a finite CW-complex then X is also a CW-complex and $\chi(X) = k \cdot \chi(Y)$.

2. Use the Mayer-Vietoris sequence to compute the homology of the space X obtained by identifying three n-discs along their boundaries.

3. Use the Mayer-Vietoris sequence to compute $H_*(\mathbb{R}P^2)$.

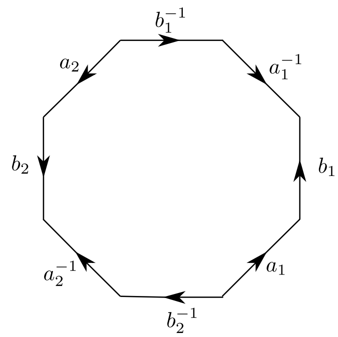
4. The Klein bottle K is the space obtained from the square I^2 by identifying opposite sides as indicated in the following picture:



   Use Mayer-Vietoris to compute $H_*(K)$.

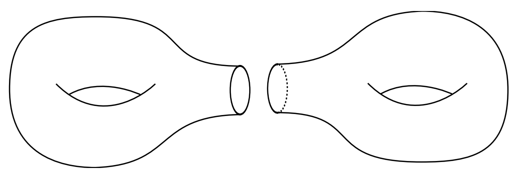
5. Consider a polygon with $4g$ edges which are grouped into g tuples, each consisting of 4 consecutive edges labelled in counterclockwise order by $a_k, b_k, a_k^{-1}, b_k^{-1}$ for $1 \leq k \leq g$ (the figure shows the case $g = 2$). By identifying the edges according to the labelling, one obtains a closed orientable surface Σ_g of genus g.

   Compute $H_*(\Sigma_g)$ using this description and the Mayer-Vietoris sequence.



6. Given two manifolds M_0, M_1 of the same dimension, one can construct their *connected sum* $M_0 \# M_1$ by cutting out the interiors of two embedded closed discs $D_0 \subset M_0, D_1 \subset M_1$, and identifying the boundaries ∂D_0 and ∂D_1 by some homeomorphism. (One doesn't need to precicely know what a manifold is in order to solve this exercise.)

   An alternative inductive construction of the orientable genus g surfaces Σ_g is as follows: Σ_1 is the torus T^2, and Σ_g is defined as $\Sigma_{g-1} \# \Sigma_1$ for $g \geq 2$. The figure shows how Σ_2 arises that way.

   Compute $H_*(\Sigma_g)$ using this description and the Mayer-Vietoris sequence.

7. Construct a cycle that represents a generator of $\widetilde{H}_n(S^n)$ for $n = 0, 1, 2$. (Start with $n = 0$, then pass to $n = 1$ using Mayer-Vietoris, and then to $n = 2$ using Mayer-Vietoris.)

8. Suppose that $X \vee Y$ be the wedge product obtained by identifying two points which are deformation retracts of neighbourhoods $U \subset X$ and $V \subset Y$. Show that $\widetilde{H}_n(X \vee Y) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y)$ for all n using Mayer-Vietoris.

9. Denote by SX the (unreduced) suspension of X, obtained from $I \times X$ by collapsing $\{0\} \times X$ and $\{1\} \times X$ each to a point. Use Mayer-Vietoris to show that $\widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X)$.