Prof. Paul Biran
ETH Zürich

Problem set 6

1. Recall that a retraction of a space X onto a subspace $A \subset X$ is a map $r : X \to A$ such that $r|_A = \mathrm{id}_A$. Prove that there exists no retraction $f : D^n \to \partial D^n$.

2. Prove the Brouwer fixed point theorem for D^n, which says that every continuous map $f : D^n \to D^n$ has a fixed point.

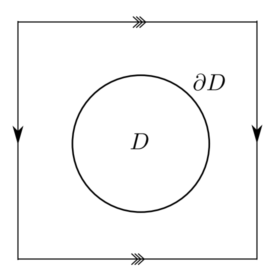
3. Prove the fundamental theorem of algebra, i.e., that every polynomial function $p : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 1$ has at least one zero. *Hint:* You might think about what p does suitably to chosen circles in $\mathbb{C}$, and about degrees.

4. Prove that there does not exist a nowhere-vanishing vector field on any even-dimensional sphere S^{2n}.

5. Prove that there exists a surjective map $S^n \to S^n$ of degree zero for any $n \geq 1$.

6. Let $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ be a map which induces an isomorphism in homology. Prove that f is surjective.

7. Consider the torus T^2 and an embedded disc $D \subset T^2$ as in the picture. Set $X = T^2 \setminus \operatorname{int} D$. Prove that X does not retract onto $\partial D \subset X$.



8. Let $p \in \mathbb{N}$, $p > 1$. The space R_p is obtained from the ball B^3 by identifying points on its boundary $\partial B^3 = S^2$ as follows:
   Given any point on the boundary $\partial B^3 = S^2$, identify it with its rotation by the angle $\frac{2\pi}{p}$ about the vertical axis.

   (a) Give a CW-complex structure on R_p.

(b) Compute the cellular homology of R_p.

9. Find a CW complex structure for the closed orientable genus g surface Σ_g and compute $H_*(\Sigma_g)$ using cellular homology. *Hint:* Recall from problem set 5 the construction of Σ_g from a $4g$-sided polygon.

10. Let $f : C_\bullet \to D_\bullet$ be a chain map between the two chain complexes $C_\bullet$ and $D_\bullet$.

    (a) Show that there is a short exact sequence of chain complexes

    $$0 \to D \to cone(f) \to C[-1] \to 0$$

    where $cone(f)_\bullet$ is the (algebraic) mapping cone of f.

    (b) Show that the connecting homomorphism of the induced long exact sequence on homology is given by the induced map $f_* : H_*(C) \to H_*(D)$.

    (c) Show that a chain map $f : C_\bullet \to D_\bullet$ is a quasi-isomorphism if and only if the mapping cone $cone(f)_\bullet$ is an acyclic chain complex.

11. Let X be a finite CW complex and suppose that $p : \widetilde{X} \to X$ is an n-sheeted covering map. Prove that $\chi(\widetilde{X}) = n\chi(X)$.

12. Show that if the closed orientable surface Σ_g is a covering space of Σ_h, then $g = n(h-1)+1$ for some n.