

### Solutions to problem set 3

1. (a) Since  $f(x) \neq x, \forall x \in S^n$ , the line segment  $(1-t)f(x) - tx, t \in [0, 1]$ , does not pass through 0. Therefore, if  $f$  has no fixed points,

$$f_t(x) := \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$$

is a well defined homotopy from  $f$  to the antipodal map  $-id$  which has degree  $\deg(-id) = (-1)^{n+1}$ . Thus  $\deg f = (-1)^{n+1}$ .

- (b) Since  $\deg f = 0 \neq (-1)^{n+1}$  it must have a fixed point  $x \in S^n$  by exercise 1a), i.e.  $f(x) = x$ . For the same reason, since  $g := (-id) \circ f$  has degree  $\deg g = \deg(-id) \cdot \deg f = 0$  there a fixed point  $y \in S^n$  of  $g$ , i.e.  $g(y) = -f(y) = y$ . Which means that  $f(y) = -y$ .
2. Let the group action be given by the homomorphism  $\rho : G \rightarrow \text{Homeo}(S^n)$ . The degree of a homoemorphism is always  $\pm 1$ . Therefore the group action determines a degree function  $d : G \rightarrow \{\pm 1\}$  given by  $d(g) := \deg \rho(g)$ . Furthermore  $d$  is a homomorphism:

$$d(hg) = \deg \rho(hg) = \deg(\rho(h) \circ \rho(g)) = \deg \rho(h) \cdot \deg \rho(g) = d(h) \cdot d(g).$$

If  $g \in G$  is a non trivial element, then  $\rho(g)$  has no fixed points as the action is free and hence (by exercise 1a)) we have  $d(g) = (-1)^{n+1}$ . So, if  $n$  is even, the kernel of  $d$  is trivial which implies that  $G$  is isomorphic to a subgroup of  $\{\pm 1\} \cong \mathbb{Z}_2$ .

3. See example 2.32 on page 137 in Hatcher's book.

4. (a) Let  $I := [0, 1]$  and let  $CS^n := (S^n \times I)/(S^n \times \{1\})$  denote the cone with base  $S^n = S^n \times \{0\} \subset CS^n$ . Notice that  $CS^n/S^n$  is the suspension of  $S^n$  and in particular  $CS^n/S^n \cong S^{n+1}$ . The map  $f$  induces  $Cf : (CS^n, S^n) \rightarrow (CS^n, S^n)$  with quotient  $Sf$ . Since  $S^n \times \{0\}$  is a deformation retract of some open neighborhood in  $CS^n$  we get (see Proposition 2.22 in Hatcher) that

$$H_{n+1}(CS^n, S^n) \cong H_{n+1}(CS^n/S^n, S^n/S^n) \cong \tilde{H}_{n+1}(CS^n/S^n) \cong \tilde{H}_{n+1}(S^{n+1}).$$

The naturality of the boundary maps in the long exact sequence of the pair  $(CS^n, S^n)$  implies that we have the following commutative diagram

$$\begin{array}{ccc} \tilde{H}_{n+1}(S^{n+1}) & \xrightarrow[\cong]{\partial_*} & \tilde{H}_n(S^n) \\ \downarrow Sf_* & & \downarrow f_* \\ \tilde{H}_{n+1}(S^{n+1}) & \xrightarrow[\cong]{\partial_*} & \tilde{H}_n(S^n) \end{array}$$

The two horizontal maps are isomorphisms since  $CS^n$  is contractible and hence  $\tilde{H}_*(CS^n) = 0$  in the long exact sequence of the pair. Therefore, if  $f_*$  is multiplication by  $d = \deg f$ , then  $Sf_*$  is also multiplication by  $d$  and hence  $\deg f = \deg Sf$ .

- (b) Given  $k \in \mathbb{Z}$  the map  $S^1 \rightarrow S^1 : z \mapsto z^k$  has degree  $k$ . Now assume that we have constructed a map  $f : S^n \rightarrow S^n$  of degree  $k$ , then (by exercise 4a)), the map  $Sf : S^{n+1} \rightarrow S^{n+1}$  has degree  $k$  as well. So, the claim follows by induction.

5. First, let  $n = 1$  and denote  $I := [0, 1]$ . Let  $g : I \rightarrow \mathbb{R}$  be a continuous map such that  $g(0) = g(1) = 0$  and  $g(1/2) = 2\pi$ . The map  $g$  induces a well defined continuous surjection  $f : I/\partial I = S^1 \rightarrow S^1 : t \mapsto e^{ig(t)}$ . By the path lifting property the map  $g$  is the unique lift of  $f$  to the universal cover  $\mathbb{R}$  of  $S^1$  starting at the point  $0 \in \mathbb{R}$ . So  $f \in p_{\#} \underbrace{\pi_1(\mathbb{R}, 0)}_{=0} \subset \pi_1(S^1, 1)$

is homotopic to a constant map (which is clearly not surjective and therefore has degree 0) and hence  $\deg f = 0$ . Here,  $p : \mathbb{R} \rightarrow S^1$  is the universal cover.

Using exercise 4a) we obtain, by repeatedly suspending the map  $f$ , a surjective map  $S^n \rightarrow S^n$  of degree 0.

For an alternative, more explicit, solution see example 2.31 in Hatcher's book.

6. For  $n = 2$  we have that  $SO(2)$  is homeomorphic to the circle  $S^1$  which is path connected. Proceeding by induction we assume that  $SO(n-1)$  is path connected. Given any  $A \in SO(n)$  it is enough to show that there is a path in  $SO(n)$  connecting  $A$  to the identity matrix  $I_n$ . This means that we need to find a continuous path taking the standard basis  $e_1, \dots, e_n$  to their image  $Ae_1, \dots, Ae_n$ . Let  $\Lambda \subset \mathbb{R}^n$  be a plane containing both  $e_1$  and  $Ae_1$ . By the path connectedness of  $SO(2)$ , we can continuously move  $e_1$  to  $Ae_1$  by a rotation  $R$  of the plane  $\Lambda$ .

It remains to continuously move  $Re_2, \dots, Re_n$  to  $Ae_2, \dots, Ae_n$  while keeping  $Ae_1$  fixed. Notice that  $Ae_1 = Re_1 \perp Re_i$  and  $Ae_1 \perp Ae_i$  for each  $2 \leq i \leq n$  since both  $R$  and  $A$  preserve angles. Hence the required motion can take place in the hyperplane  $\mathbb{R}^{n-1}$  of vectors orthogonal to  $Ae_1$ , where it exists by the assumption that  $SO(n-1)$  is path connected.

Concatenating the two motions gives a path in  $SO(n)$  from  $I_n$  to  $A$  and thus  $SO(n)$  is path connected.