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Solutions to problem set 4

1. $\mathbb{R}P^n$ has a CW complex structure with exactly one k-cell a_k for every $k = 0, \dots, n$. We have also seen that the degree of the map $p_{a_{k-1}} f_{\partial a_k} : S^{k-1} \to S^{k-1}$ is

$$\deg(p_{a_{k-1}} f_{\partial a_k}) = 1 + (-1)^k \begin{cases} 0, & k \text{ odd}, \\ 2, & k \text{ even} \end{cases}$$

The corresponding $\mathbb{Z}_2$-degree are obtained by reducing modulo 2, and hence

$$\deg_{\mathbb{Z}_2}(p_{a_{k-1}} f_{\partial a_k}) = 0$$

for all $k = 1, \dots, n$. The cellular chain complex with $\mathbb{Z}_2$-coefficients is therefore

$$0 \to \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \dots \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} 0$$

with non-zero chain groups exactly in degrees $0, \dots, n$, and thus we obtain

$$H_k(\mathbb{R}P^n; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & k = 0, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that in general $H_*(\mathbb{R}P^n; \mathbb{Z}_2)$ has more non-vanishing components than $H_*(\mathbb{R}P^n; \mathbb{Z})$; in particular, $H_*(\mathbb{R}P^n; \mathbb{Z}_2) \neq H_*(\mathbb{R}P^n; \mathbb{Z}) \otimes \mathbb{Z}_2$.

2. We view $T^3 = I^3/\sim$ as the quotient space of the cube I^3 under the relation that identifies opposite faces of the boundary. From this description, one sees that T^3 has a CW complex structure with one 0-cell a (any of the corner points—note that these get identified under $I^3 \to T^3$), three 1-cells b_1, b_2, b_3 (the line segments on the coordinate axes), three 2-cells c_1, c_2, c_3 (the squares in the coordinate planes), and one 3-cell d (all of I^3); in all these cases the attaching maps is given by restriction of the quotient map $I^3 \to T^3$.

The corresponding cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z}^3 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z} \to 0$$

with linear maps ∂_i which we now compute. We have $\partial_1 = 0$ since the attaching maps $f_{b_i} : I \to (T^3)^{(0)} = \{a\}$ take both boundary points $0, 1 \in I$ to the same point (cf. the remark in Bredon after Theorem 10.3). We also have $\partial_2 = 0$, since all maps $p_{b_i} f_{\partial c_j} : \partial I^2 \to S^1$ have degree 0 (by the same argument as for the standard CW complex structure of the 2-torus; see Bredon example 10.5).

As for ∂_3, consider any of the maps $p_{c_i} f_{\partial d} : \partial I^3 \to S^2$. Note that there are two opposite faces of ∂I^3 in whose interiors this map restricts to a homeomorphism, and that the map collapes the rest of ∂I^3 to a point in S^2. The degree of $p_{c_i} f_{\partial d}$ is hence the sum of the two local degrees at any two points q, q' in the two first-mentioned faces which get mapped to the same point in T^3. Now note that the restrictions of $p_{c_i} f_{\partial d}$ to these faces are obtained from one another by precomposing with an orientation-*reversing* map (for orientations induced from an orientation of ∂I^3); therefore the sum of these local degrees vanishes. It follows that also $\partial_3 = 0$.

Summing up, we obtain

$$H_i(T^3) \cong \begin{cases} \mathbb{Z}, & i = 0, 3, \\ \mathbb{Z}^3, & i = 1, 2. \end{cases}$$

3. (a) One possible CW complex structure has two 0-cells a_1, a_2 (the north and south poles), two 1-cells b_1, b_2 (the line segment mentioned in the description of X and another segment on the sphere connecting the poles), and one 2-cell c. We then have

$$\deg(p_{a_2} f_{\partial b_j}) = 1, \quad \deg(p_{a_1} f_{\partial b_j}) = -1$$

for $j = 1, 2$, supposing that the attaching maps $f_{b_j} : I \to X^{(0)}$ are such that both map $0 \in \partial I$ to a_1 and $1 \in \partial I$ to a_2 (cf. the remark in Bredon after Theorem 10.3). Moreover, we have

$$\deg(p_{b_j} f_{\partial c}) = 0$$

for $j = 1, 2$, as both maps $p_{b_j} f_{\partial c}$ are null-homotopic. The cellular chain complex is therefore

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{\partial_1} \mathbb{Z}^2 \to 0, \quad \partial_1 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^2.$$

Both the kernel and the cokernel of ∂_1 are 1-dimensional, and therefore

$$H_k(X) \cong \begin{cases} \mathbb{Z}, & k = 0, 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

(Note that there is an even simpler CW complex structure for X with exactly one k-cell for $k = 0, 1, 2$.)

(b) $X \simeq S^2 \vee S^1$ implies $\widetilde{H}_*(X) = \widetilde{H}_*(S^2 \vee S^1) \cong \widetilde{H}_*(S^2) \oplus \widetilde{H}_*(S^1)$; hence $\widetilde{H}_2(X) = \widetilde{H}_1(X) = \mathbb{Z}$ and $\widetilde{H}_0(X) = 0$, from which the result above follows by the definition of reduced homology.

Alternatively: Excising a neighbourhood of the point joining the two spheres yields $\widetilde{H}_*(X) \cong H_*(D^2, \partial D^2) \oplus H_*(I, \partial I)$ from which the result above again follows easily.

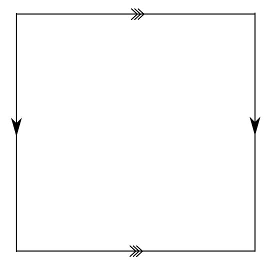
4. We assume wlog that p and q are coprime (otherwise divide by their greatest common divisor), which implies that there exist integers a, b such that $ap - bq = 1$. Hence the matrix

$$\Psi = \begin{pmatrix} a & q \\ b & p \end{pmatrix}$$

lies in $SL(2, \mathbb{Z})$ and therefore induces a homeomorphism $\psi : T^2 \to T^2$ of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Note that $\Psi^{-1} \in SL(2, \mathbb{Z})$ takes the line given by $px = qy$ to the line given by $x = 0$, because Ψ takes $(0, 1)$ to (q, p) (and these vectors generate the two lines). Therefore ψ^{-1} takes C to the curve C' that's the image of $x = 0$ under $\mathbb{R}^2 \to T^2$ and which is the 1-cell of the standard CW complex structure on T^2. Thus T^2/C has a CW complex structure with one cell a_k in dimensions $k = 0, 1, 2$, and the corresponding cellular differential vanishes (by the same reasons as for T^2). Therefore

$$H_k(T^2/C) \cong \begin{cases} \mathbb{Z}, & k = 0, 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

5. We view $S^1 \times S^1$ as $I^2/\sim$, the quotient obtained by identifying opposite points on the boundary of ∂I^2 as indicated in the figure below. We endow it with the corresponding obvious CW complex structure with one 0-cell, two 1-cells, and one 2-cell and arrange this to be such that the subspace $S^1 \vee S^1$ that gets collapsed is the union of the two closed 1-cells. Moreover, we equip S^2 with the obvious CW complex structure with one 0-cell and one 2-cell, arranging that the 0-cell is the point to which $S^1 \vee S^1$ gets collapsed.



Our quotient map $g : S^1 \times S^1 \to S^2$ is cellular in this identification. Denoting the 2-cell of $S^1 \times S^1$ by σ and the 2-cell of S^2 by τ, the map $g_\Delta : C_*(S^1 \times S^1) \to C_*(S^2)$ induced by g on cellular chains takes $\sigma \mapsto g_\Delta(\sigma) = \tau$ because $\deg(g_{\tau,\sigma}) = 1$ for the relevant map $g_{\tau,\sigma} : S^2 \to S^2$ (see Bredon chapter IV. 11). The induced map $g_* : H_2(S^1 \times S^1) \times H_2(S^2)$ is hence the identity, and therefore g is not null-homotopic.

Let now $f : S^2 \to S^1 \times S^1$ be a map in the other direction. Consider the covering map $q : \mathbb{R}^2 \to S^1 \times S^1$ (obtained by identifying $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$). As $\pi_1(S^2)$ is trivial, f can be lifted to a map to $\mathbb{R}^2$, i.e., there exists a map $\widetilde{f} : S^2 \to \mathbb{R}^2$ such that $q \circ \widetilde{f} = f$. Since $\mathbb{R}^2$ is contractible, $\widetilde{f}$ is null-homotopic, and hence so is f.

6. As discussed in class, $\mathbb{R}P^n$ has a CW complex structure with exactly one k-cell for every $k = 0, \ldots, n$. Therefore $\mathbb{R}P^n/\mathbb{R}P^m$ has a CW complex structure with one 0-cell a_0 and one k-cell a_k for every $k = m+1, \ldots, n$. As in the case $\mathbb{R}P^n$, we have

$$\deg(p_{a_{k-1}} f_{\partial a_k}) = 1 + (-1)^k \begin{cases} 0, & k \text{ odd}, \\ 2, & k \text{ even}. \end{cases}$$

Thus the cellular chain complex $C_*(\mathbb{R}P^n/\mathbb{R}P^m)$ has one copy of $\mathbb{Z}$ in degrees $k = 0$ and $k = m+1, \ldots, n$, and the cellular differential $C_k(\mathbb{R}P^n/\mathbb{R}P^m) \to C_{k-1}(\mathbb{R}P^n/\mathbb{R}P^m)$ is $1 + (-1)^k$ for all $k = m+2, \ldots, n$ and vanishes in all other cases. The homology is therefore

$$H_k(\mathbb{R}P^n/\mathbb{R}P^m) \cong \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}, & k = m+1 \text{ (if } m+1 \text{ is even)}, \\ \mathbb{Z}, & k = n \text{ (if } n \text{ is odd)}, \\ \mathbb{Z}_2, & m+1 \leq k < n \text{ and } k \text{ odd}, \\ 0, & \text{otherwise}. \end{cases}$$