Final — Math 317 Fall 2004

The final is a 3 hour closed book exam. You may not use class notes. Return to my mailbox on the third floor by 5PM on January 17.

I. Let $D \subset \mathbb{C}$ be the open disk of radius 2 at the origin. Let

$$f : D \to \mathbb{C}$$

be an analytic function, and let

$$\sum_{n=0}^{\infty} \alpha_n (z-1)^n$$

be the Taylor series of f at $1 \in D$.

(i) Must the sum $\sum_{n=0}^{\infty} \alpha_n (-1)^n$ converge?

(ii) Must the sum $\sum_{n=0}^{\infty} \alpha_n (4/7)^n$ converge?

(iii) Must the sum $\sum_{n=0}^{\infty} n^2 \alpha_n (1/3)^n$ converge?

In each case, either prove convergence or find a counterexample.

II. Consider the meromorphic function

$$f(z) = \frac{z+2}{z^2-4z+3}.$$

Let $C \subset \mathbb{C}$ be a circle in the complex plane which does not pass through any of the zeros or poles of f.

(i) Find the zeros and poles of f. At each pole, find the residue.

(ii) Calculate all possible values of the contour integral

$$\int_C f(z)dz$$

taken with counter-clockwise orientation.

(iii) Calculate all possible values of the contour integral

$$\int_C \frac{f'(z)}{f(z)}dz$$

taken with counter-clockwise orientation.

III. Calculate the following integrals:

(i) $\int_0^\infty \frac{\cos(x)}{x^2+1} dx$,

(ii) $\int_0^\infty \frac{x^2}{(x^2+1)(x^2+9)} dx$.

IV. An analytic function $f : \mathbb{C} \to \mathbb{C}$ is *periodic* if

$$f(z+1) = f(z)$$

for every $z \in \mathbb{C}$.

(i) Find an example of an analytic function $f : \mathbb{C} \to \mathbb{C}$ that is periodic and not identically zero.

(ii) If $f : \mathbb{C} \to \mathbb{C}$ is analytic and f^2 is periodic, must f also be periodic? Answer with proof.

(iii) If $f : \mathbb{C} \to \mathbb{C}$ is a periodic analytic function, can $f(1/z)$ have a pole of finite order at the origin? Answer with proof.

V. Let $D \subset \mathbb{C}$ be the open disk of radius 1 at the origin. Let Q denote the first quadrant of the complex plane

$$Q = \{x + iy \mid x > 0, \ y > 0\} \subset \mathbb{C}.$$

(i) Find an invertible analytic map $f : Q \to D$.

Note: f is invertible if $f^{-1} : D \to Q$ is well-defined. In other words, f is invertible if f is 1-to-1 and onto.

(ii) Does there exist a nonconstant analytic function $g : \mathbb{C} \to Q$?

VI. Let Q be the first quadrant of the complex plane as before,

$$Q = \{x + iy \mid x > 0,\ y > 0\} \subset \mathbb{C}.$$

The boundary B of Q consists of the positive halves of the x and y axes,

$$B = \{x + 0i \mid x \geq 0\}\ \cup \{0 + iy \mid y \geq 0\}.$$

(i) Find a harmonic function $u : Q \to \mathbb{R}$ not identically zero such that u has boundary value 0 on B.

(ii) Find a harmonic function $u : Q \to \mathbb{R}$ with boundary value 0 on

$$\{x + i0 \mid x > 0\}$$

and boundary value 1 on

$$\{0 + iy \mid y > 0\}.$$

VII. Calculate the contour integral $\int_C f(z)dz$ in the following cases:

(i) C is the path parametrized by $\gamma : [0, 1] \to \mathbb{C}$ where

$$\gamma(t) = t^4 e^t + ie^{-t}$$

and $f(z) = 3$.

(ii) C is the path parametrized by $\gamma : [0, 3\pi] \to \mathbb{C}$ where

$$\gamma(t) = \cos(t) + i\sin(t)$$

and $f(z) = 3 + \frac{1}{z}$.

(iii) C is the path parametrized by $\gamma : [0, 6] \to \mathbb{C}$ where

$$\gamma(t) = t^2 - 6t + i(t^3 - 36t)$$

and $f(z) = \exp(z + e^z)$.

VIII. Let $f(z) = \frac{z^3}{(z-2)(z+2)}$.

(i) Find the Taylor series of f at the origin.

(ii) Does there exist an analytic function g defined in a neighborhood of the origin satisfying $g^2 = f$? Answer with proof.

(iii) Does there exist an analytic function g defined in a neighborhood of the origin satisfying $g^2 = df/dz$? Answer with proof.