SOLUTIONS TO SELECTED EXERCISES: SHEET 1

Exercise 1

Write $v = (v_1, \dots, v_d)$, then the map $g \mapsto g \cdot v$ is explicitely given by:

$$(g_{i,j}) \mapsto \left(\sum_{j=1}^{d} g_{1j} v_j, \dots, \sum_{j=1}^{d} g_{dj} v_j \right)$$

This assignment clearly defines a continuous map $R^{d^2} \to R^d$ and hence restricts to a continuous map on $\mathrm{SL}_d(R)$. In particular $G_v \subseteq \mathrm{SL}_d(R)$ is a closed subset of $\mathrm{SL}_d(R)$ and thus locally compact. It is clear, that it forms a group and the restrictions of the (continuous) group operations on $\mathrm{SL}_d(\mathbb{R})$ yield continuous group operations on G_v.

Exercise 2

It is clear, that Homeo(X) forms a group: composition of functions is associative, id is a neutral element for the composition and inverses exist by the definition of a homeomorphism. So it only remains to prove continuity of these operations. As Homeo(X) is a metric space, it suffices to show that the operations preserve limits of sequences.

Assume that $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are Cauchy sequences in Homeo(X) with limits $f, g \in$ Homeo(X) respectively. The latter exist as $\mathscr{C}(X, X)$ is complete with respect to the uniform topology. We check first that inversion in Homeo(X) is continuous with respect to the topology of uniform convergence, that is $(f_n^{-1})_{n\in\mathbb{N}}$ is Cauchy and the limit is f^{-1}. Let $x \in X$ arbitrary and denote by $y \in X$ the preimage of x under f_n, then:

$$d(f^{-1}x, f_n^{-1}x) = d(f^{-1}f_n y, f_n^{-1}f_n y) = d(f^{-1}f_n y, f^{-1}fy)$$

Now let $\epsilon > 0$ arbitrary. As f^{-1} is uniformly continuous, there is some $\delta > 0$ such that:

$$d(x, y) < \delta \Rightarrow d(f^{-1}x, f^{-1}y) < \epsilon$$

As $f_n \to f$ uniformly, there is $n_0 \in \mathbb{N}$ such that:

$$n \geq n_0 \Rightarrow d(f_n y, fy) < \delta \quad \forall y \in X$$

Given $n \in \mathbb{N}$, denote by $y_{n,x} \in X$ the unique element satisfying $f_n y_{n,x} = x$, and plug in the above estimates to obtain:

$$n \geq n_0 \Rightarrow d(f^{-1}x, f_n^{-1}x) = d(f^{-1}f_n y_{n,x}, f^{-1}fy_{n,x}) < \epsilon$$

as $d(f_n y_{n,x}, fy_{n,x}) < \delta$ by assumption. This proves that $f_n^{-1} \to f^{-1}$ uniformly and in particular it shows that inversion is continuous. For continuity of coposition, let $\delta > 0$ such that $d(x, y) < \delta$ implies $d(fx, fy) < \frac{\epsilon}{2}$ and choose $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $x \in X$ hold $d(gx, g_n x) < \delta$ and $d(fx, f_n x) < \frac{\epsilon}{2}$, then:

$$n \geq n_0 \Rightarrow d(f \circ gx, f_n \circ g_n x) \leq d(f \circ gx, f \circ g_n x) + d(f \circ g_n x, f_n \circ g_n x) < \epsilon$$

This proves that Homeo(X) is a topological group.

Exercise 3

Part a.

3.a.1: As $|||\cdot|||$ and $\|\cdot\|$ induce the same topology, for $(x_n)_{n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$ holds $\|x_n\| \to 0$ if and only if $|||x||| \to 0$, and in particular $|||x|||^n = |||x^n||| \to 0$ if and only if $\|x\|^n = \|x^n\| \to 0$.

3.a.2: Let $\|x\| \in (0,1)$. If $y \in \mathbb{Q} \setminus \{0\}$ arbitrary, then $\|y\| = \|x\|^\alpha$ for some $\alpha \in \mathbb{R}$. Let $p \in \mathbb{Z}, q \in \mathbb{N}$ such that $\alpha < \frac{p}{q}$, then $\|y\| > \|x\|^{\frac{p}{q}}$ and hence $\|x^p/y^q\| < 1$. As of the preceding result it follows that $|||x^p/y^p||| < 1$ and hence $|||y||| > |||x|||^{\frac{p}{q}}$. Similarly for $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $\alpha > \frac{p}{1}$, we find:

$$\|y\| < \|x\|^{\frac{p}{q}} \Rightarrow \|y^q/x^p\| < 1 \Leftrightarrow |||y^q/x^p||| < 1 \Rightarrow |||y||| < |||x|||^{\frac{p}{q}}$$

Hence follows $|||y||| = |||x|||^\alpha$ as $t \mapsto |||x|||^t$ is continuous. If we let $\beta \in \mathbb{R}$ such that $|||x|||^\beta = \|x\|$, we obtain:

$$\|y\| = \|x\|^\alpha = |||x|||^{\alpha\beta} = |||y|||^\beta$$

Part b. Assume that $\alpha \in (0,1)$, then the map $f : x \mapsto x^\alpha$ on $(0,\infty)$ satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}x} f = \alpha x^{\alpha-1} > 0$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f = \alpha(\alpha-1)x^{\alpha-2} < 0$$

Thus f is monotonic, concave and in particular subadditive. It follows that:

$$|x+y|^\alpha = f\big(\,|x+y|\,\big) \le f\big(\,|x|+|y|\,\big) \le f\big(\,|x|\,\big) + f\big(\,|y|\,\big) = |x|^\alpha + |y|^\alpha$$

If $\alpha > 1$, $\left(\frac{1}{2}\right)^\alpha < \frac{1}{2}$, thus for $x = y = \frac{1}{2}$:

$$|x|^\alpha + |y|^\alpha < \frac{1}{2} + \frac{1}{2} = 1 = |x+y|^\alpha$$

The cases $\alpha \in \{0,1\}$ don't require a proof.

Part c. Multiplicativity of $\|\cdot\|$ implies that $\left\|\frac{p}{q}\right\| = \frac{\|p\|}{\|q\|}$ for all $p \in \mathbb{Z}, q \in \mathbb{N}$. This holds as $\|1\| = \|1 \cdot 1\| = \|1\| \cdot \|1\|$ implies $\|1\| = 1$ and thus $\left\|\frac{1}{q}\right\| \|q\| = \|1\|$.

Case 1: Assume that $\|n\| \le 1$ for all $n \in \mathbb{Z}$. By the preceding argument either $\|\cdot\|$ is trivial or there exists some $n \in \mathbb{Z}$ such that $\|n\| < 1$. As $1 = \|1\| = \|-1\|\,\|-1\|$, we can assume w.l.o.g. that $n > 0$. Let n minimal with this property, then n is a prime. Otherwise the fundamental theorem of arithmetic would imply the existence of a prime $p < n$ and $0 < n' < n$ such that:

$$1 > \|n\| = \|pn'\| = \|p\|\,\|n'\|$$

contradicting minimality of n. Let $q \in \mathbb{N}$ be a prime and assume that $q \neq n$. We claim that $\|q\| = 1$. Assume otherwise, i.e. assume that $\|q\| < 1$. As $\gcd(q,n) = 1$, for all $l, m \in \mathbb{N}$ we can find $r, s \in \mathbb{Z}$ such that $rq^m + sn^l = 1$. Hence:

$$1 = \|1\| \le \|r\|\,\|q\|^m + \|s\|\,\|n\|^l \le \|q\|^m + \|n\|^l \overset{l,m\to\infty}{\longrightarrow} 0$$

Thus the fundamental theorem of arithmetic implies that:

$$\|r\| = \|n\|^{\nu_n(r)} \quad \forall r \in \mathbb{Z} \setminus \{0\}$$

where $\nu_n(r) = \max\left\{k \in \mathbb{Z}; n^k | r\right\}$, proving that $\|\cdot\| = |\cdot|_n^{-\alpha}$ with:

$$\alpha = -\frac{\log \|n\|}{\log n} < 0$$

Case 2: Fix $n \in \mathbb{N}$ minimal such that $\|n\| > 1$. Let $\alpha \in \mathbb{R}$ such that $\|n\| = n^\alpha$. If $m \in \mathbb{N}$ arbitrary, there exists some $r \geq 0$ together with a unique tuple $(a_0, \dots, a_r) \in \{0, \dots, n-1\}$ such that $a_r \neq 0$ and:

$$m = \sum_{k=0}^{r} a_k n^k$$

Minimality of n implies that:

$$\begin{aligned} \|m\| \leq & \sum_{k=0}^{r} \|a_k\| \, \|n\|^k = \sum_{k=0}^{r} n^{\alpha k} \\ = & n^{\alpha r} \sum_{k=0}^{r} n^{\alpha(k-r)} \leq C m^\alpha \end{aligned}$$

where $C = \frac{1}{1-\frac{1}{n^\alpha}}$ is independent of m. Thus $\|m\| = \left\|m^l\right\|^{\frac{1}{l}} \leq C^{\frac{1}{l}} m^\alpha$ for all $l \in \mathbb{N}$ and $\|m\| \leq m^\alpha$.

We produce the same estimate in the opposite direction as follows: Given $m \in \mathbb{N}$, let $s \in \mathbb{N}_0$ such that $n^{s+1} > m \geq n^s$, then – using the preceding estimate – $\left\|n^{s+1} - m\right\| \leq (n^{s+1} - m)^\alpha$ implies:

$$\begin{aligned} \|m\| \geq & \left\|n^{s+1}\right\| - \left\|n^{s+1} - m\right\| \\ \geq & n^{\alpha(s+1)} - (n^{s+1} - n^s)^\alpha \\ = & n^{\alpha(s+1)} \frac{1}{n^\alpha} \geq C m^\alpha \end{aligned}$$

with $C = n^{-\alpha}$. Again multiplicativity implies that $\|m\| \geq C^{\frac{1}{l}} m^\alpha$ for all $l \in \mathbb{N}$ and thus $\|m\| \geq m^\alpha$. This proves equality.

As $m \in \mathbb{N}$ was arbitrary, the initial exposition implies that $\left\|\frac{p}{q}\right\| = \left|\frac{p}{1}\right|^\alpha$ for all $p \in \mathbb{Z}, q \in \mathbb{N}$. It follows from part (b) and the assumption $\|n\| > 1$ that $\alpha \in (0, 1]$.