SOLUTIONS TO SELECTED EXERCISES: SHEET 2

Exercise 1

a: Let $g, h \in H$. Then there exists $n \in \mathbb{N}$ such that $g, h \in (U \cup U^{-1})^n$, so that $gh \in (U \cup U^{-1})^{2n}$. If $h \in (U \cup U^{-1})^n$, we find $h_1, \ldots, h_n \in U \cup U^{-1}$ such that $h = h_1 \cdots h_n$. Clearly $h_1^{-1}, \ldots, h_n^{-1} \in U \cup U^{-1}$, $h^{-1} = h_n^{-1} \cdots h_1^{-1} \in (U \cup U^{-1})^n$, so H is a subgroup of G. In order to see that $H \subseteq G$ is an open subset, we note that by continuity of inversion, $U \cup U^{-1}$ contains an open neighbourhood of $1 \in G$, thus $g(U \cup U^{-1}) \subseteq H$ contains an open neighbourhood of g proving openness.

b: Let $V := \mathring{U}$ and $W := (V \cap V^{-1})$, then W is open as inversion is an isomorphism and $W^n \subseteq U^n$ for all $n \geq 0$ and hence it suffices to show $G = \bigcup_{n \geq 1} W^n$. $W^n = \bigcup_{g \in W^{n-1}} gW$ is open, thus so is $H := \bigcup_{n \geq 1} W^n$. As $W^{-1} = W$. Part a implies that $H \leq G$ is an open and thus closed subgroup of G. As G is connected and $1 \in H$, it follows $G = H$.

Exercise 2

a: Let $\phi \in \mathrm{Aut}(\mathbb{R}^n)$, then ϕ is in particular additive and thus $\phi(nv) = n\phi(v)$ for all $v \in \mathbb{R}^n$, for all $n \in \mathbb{Z}$. Let $m \in \mathbb{Z}$, $n \in \mathbb{N}$ and $q = \frac{m}{n} \in \mathbb{Q}$, then:

$$n\phi(qv) = \phi(nqv) = \phi(mv) = m\phi(v) \Rightarrow \phi(q)\phi(w) = q\phi(v)$$

so ϕ is $\mathbb{Q}$-linear. $\mathbb{R}$-linearity follows from continuity of ϕ and thus $\phi \in \mathrm{End}_{\mathbb{R}}()(\mathbb{R}^n)$. As ϕ is invertible, any choice of basis realizes ϕ as an element in $\mathrm{GL}_n(\mathbb{R})$. It is clear that for such a choice of a basis, any $g \in \mathrm{GL}_n(\mathbb{R})$ defines an element in $\mathrm{Aut}(\mathbb{R}^n)$ and that the correspondence is 1-1 and obeys the various group structures (on $\mathrm{Aut}(G)$ and $\mathrm{GL}_n(\mathbb{R})$).

b: The n-dimensional Lebesgue measure λ_n on $\mathbb{R}^n$ clearly is a Haar measure for $\mathbb{R}^n$: it is translation invariant and:

$$\lambda_n\big(B_r(v)\big) = \frac{(\sqrt{\pi}r)^n}{\Gamma(\frac{n}{2}+1)} \in (0, \infty) \quad \forall r > 0 \forall v \in \mathbb{R}^n$$

showing that it is positive on open and finite on compact subsets of $\mathbb{R}^n$. Let $f \in \mathscr{C}_c(\mathbb{R}^n)$, $g \in \mathrm{GL}_n(\mathbb{R})$, then:

$$\begin{aligned}\int_{\mathbb{R}^n} f(gv)\,\mathrm{d}\lambda_n(v) &= \frac{1}{|\det g|}\int_{\mathbb{R}^n} f(gv)\,|\det g|\,\mathrm{d}\lambda_n(v)\\ &= |\det g|^{-1}\int_{\mathbb{R}^n} f(v)\,\mathrm{d}\lambda_n(v)\end{aligned}$$

As any Borel measure on $\mathbb{R}^n$ is uniquely determined by its values on $\mathscr{C}_c(\mathbb{R}^n)$, it follows $g_*\lambda_n = |\det g|^{-1}\lambda_n$ and hence the claim.

c: Using Zorn's lemma, construct a $\mathbb{Q}$-basis of $\mathbb{R}$ containing 1. Denote this basis by $\{x_i; i \in I\}$ for any infinite index set I containing 0 such that $x_0 = 1$ (I is infinite as otherwise $\mathbb{R}$ would be algebraic over $\mathbb{Q}$). Fix $i, j \in I \setminus \{0\}$ such that $i \neq j$ and define a linear map $\phi : \mathbb{R} \to \mathbb{R}$ by $\mathbb{Q}$-linear extension of:

$$\forall k \in I : \phi(x_k) = \begin{cases} x_j & \text{if } k = i \\ x_i & \text{if } k = j \\ x_k & \text{else}\end{cases}$$

Let $(q_n)_{n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$ Cauchy such that $\lim_{n\to\infty} q_n = x_i$, then:

$$\lim_{n\to\infty} \phi(q_n) = \lim_{n\to\infty} q_n = x_i \neq x_j = \phi(x_i) = \phi(\lim_{n\to\infty} q_n)$$

Exercise 3

a: As $\mathrm{GL}_n(\mathbb{R}) = \det^{-1}\big(\mathbb{R}\setminus\{0\}\big)$ is open in $\mathbb{R}^{n^2}$, $\lambda_{n^2}\big|_{\mathrm{GL}_n(\mathbb{R})}$ assigns non-zero measure to non-empty open and finite measure to compact subsets of $\mathrm{GL}_n(\mathbb{R})$ (if $K \subseteq \mathrm{GL}_n(\mathbb{R})$ is compact in $\mathrm{GL}_n(\mathbb{R})$ and $\mathcal{U}$ an open cover of K in $\mathbb{R}^{n^2}$, then $\mathcal{U} \cap \mathrm{GL}_n(\mathbb{R}) := \{U \cap \mathrm{GL}_n(\mathbb{R}); U \in \mathcal{U}\}$ is an open cover of K in $\mathrm{GL}_n(\mathbb{R})$, thus it admits a finite subcover and hence so does $\mathcal{U}$). As det is continuous and does not vanish on $\mathrm{GL}_n(\mathbb{R})$, the above also holds for $\mathrm{d}m(g) := |\det g|^{-n}\,\mathrm{d}\lambda_{n^2}(g)$. It remains to show that m is invariant. To this end we note that for $g \in \mathrm{GL}_n(\mathbb{R})$, if $g = (g_1,\dots,g_n)$ and $h \in \mathrm{GL}_n(\mathbb{R})$, then:

$$hg = (hg_1,\dots,hg_2) \quad \forall g \in \mathrm{Mat}_n(\mathbb{R})$$

so that the left-action of h on $\mathrm{GL}_n(\mathbb{R})$ can be viewed as a restriction of a diagonal matrix $\mathrm{diag}(h,\dots,h) \in \mathbb{R}^{n^2\times n^2}$ acting on a subset of $\mathbb{R}^{n^2}$. Let $f \in \mathscr{C}_c\big(\mathrm{GL}_n(\mathbb{R})\big)$, then:

$$\begin{aligned}
&\int_{\mathbb{R}^{n^2}} \mathbb{1}_{\mathrm{GL}_n(\mathbb{R})}(g) f(hg)\,|\det g|^{-n}\,\mathrm{d}\lambda_{n^2}(g) \\
&\qquad= \int_{\mathbb{R}^{n^2}} \mathbb{1}_{h\,\mathrm{GL}_n(\mathbb{R})}(hg) f(hg)\,|\det hg|^{-n}\,|\det h|^{n}\,\mathrm{d}\lambda_{n^2}(g) \\
&\qquad= \int_{\mathbb{R}^{n^2}} \mathbb{1}_{\mathrm{GL}_n(\mathbb{R})}(hg) f(hg)\,|\det hg|^{-n}\,|\det h|^{n}\,\mathrm{d}\lambda_{n^2}(g) \\
&\qquad= \int_{\mathbb{R}^{n^2}} \mathbb{1}_{\mathrm{GL}_n(\mathbb{R})}(g) f(g)\,|\det g|^{-n}\,\mathrm{d}\lambda_{n^2}(g)
\end{aligned}$$

where in the end we used the substitution formula for the map $\mathrm{diag}(h,\dots,h)$. This proves that m is a left Haar measure on $\mathrm{GL}_n(\mathbb{R})$. The measure is also right-invariant, because the map:

$$g \mapsto \begin{pmatrix} g_1 h \\ \vdots \\ g_n h \end{pmatrix}$$

does also have Jacobian $|\det h|^n$ (for example because $gh = {}^t({}^t h\,{}^t g)$ and the Jacobian of transposition – being an idempotent map – is equal to 1). Thus $\mathrm{GL}_n(\mathbb{R})$ is unimodular.

Exercise 4

If G is assumed to be locally compact, σ-compact, metric as in the lecture, then the statement follows from Lebesgue's dominated convergence theorem (and its application to parameter dependent integrals). This argument can be generalized to locally compact spaces (c.f. [3, p. 46]) but we will provide here the argument using the regularity properties of the Haar measure: given a Haar measure m_G on G, if G is σ-compact, then the Riesz representation theorem tells us that m_G is a Radon measure, i.e. inner and outer regular.[1]

[1] If G is not σ-compact, we can choose m_G to be the restriction of a unique Radon measure on a complete σ-algebra containing all the Borel sets.

Fix $\epsilon > 0$ and any open neighbourhood $O \subseteq G$ of $1 \in G$ such that $m_G(O) < \infty$. Choose $K \subseteq O \subseteq G$ a compact neighbourhood of the identity $1 \in G$ such that $m_G(K)(1+\epsilon) > m_G(O)$, which exists by inner regularity, i.e.:

$$m_G(O) = \sup\{m_G(K); K \subseteq O \text{ compact}\}$$

For all $x \in K$ there is an open neighbourhood $V_x \subseteq G$ of $1 \in G$ such that $xV_x \subseteq O$ by continuity. Assume that U_x is chosen as a symmetric neighbourhood of the identity satisfying $U_xU_x \subseteq V_x$. As K is compact, there exists a finite subset $\{x_1, \dots, x_n\} \subseteq K$ such that $K \subseteq \bigcup_i x_iU_{x_i}$. Let $U = \bigcap_i U_{x_i}$. We claim that $KU \subseteq O$. If $y \in K$, then there is some i such that $y \in x_iU_{x_i}$ and hence:

$$yU \subseteq yU_{x_i} \subseteq x_iU_{x_i}U_{x_i} \subseteq x_iV_{x_i} \subseteq O$$

This proves that $\Delta_G(g) < 1+\epsilon$ for all $g \in U$ as follows:

$$\begin{aligned} g \in U \Rightarrow & Kg \subseteq O \Rightarrow m_G(Kg) \leq m_G(O) \\ \Rightarrow & \Delta_G(g) = \frac{m_G(Kg)}{m_G(K)} < (1+\epsilon)\frac{m_G(Kg)}{m_G(O)} < 1+\epsilon \end{aligned}$$

As the U_x were all symmetric, so is U. Hence if $g \in U$, then so is g^{-1} and thus by the above holds $\Delta_G(g^{-1}) < 1+\epsilon$. As Δ_G is a homomorphism into the positive real numbers:

$$\frac{1}{\Delta_G(g)} = \Delta_G(g^{-1}) < 1+\epsilon \Rightarrow \frac{1}{1+\epsilon} < \Delta_G(g)$$

and as $g \in U$ was arbitrary, $(1+\epsilon)^{-1} < \Delta_G(g) < 1+\epsilon$ for all $g \in U$. As ϵ was arbitrary, this proves continuity at $1 \in G$.

Exercise 7

Let $x \in \overline{H}$. As G is clearly first countable, we find $(x_n)_{n\in\mathbb{N}} \in H^N$ such that $x = \lim_{n\to\infty}$. Let $V \subseteq W \subseteq \overline{W} \subseteq U$ open neighbourhoods of $1 \in G$ with compact closure and assume that $\psi : U \to (-1,1)^{\dim G}$ is a chart as in the definition of a regular manifold. Assume furthermore that V is symmetric and $VV \subseteq W$. By assumption, there is $N \geq 1$ such that $x_n \in xV$ for all $n \geq N$, thus $x_N^{-1}x_n \in W$ for all $n \geq N$: $x_N \in xV \Rightarrow x_N^{-1} \in V^{-1}x^{-1} = Vx^{-1}$ and thus $x_N^{-1}x_n \in H \cap VV \subseteq H \cap \overline{W}$. We note that $H \cap \overline{W}$ is compact by the choice of $U - \psi(\overline{W}) \subseteq (-1,1)^{\dim G}$ is is compact, thus so is $\psi(\overline{W}) \cap \{0\}^{\dim G - \dim H} \times (-1,1)^{\dim H}$. But $x_N^{-1}x_n$ is convergent, thus has a limit y in $H \cap \overline{W}$, hence $x_Ny = x \in H$.

Exercise 8

a: Define:

$$\Phi(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) := \begin{pmatrix} a - b\mathbf{i} & c - d\mathbf{i} \\ -c - d\mathbf{i} & a + b\mathbf{i} \end{pmatrix}$$

Φ is clearly $\mathbb{R}$-linear. In order to show that Φ is a homomorphism of rings, it suffices to show that Φ obeys the product on the generators $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. This works and is supertedious to tex. It is clear that the map is a bijection and hence the claim follows. For convenience, we write down the image of the generators under Φ:

$$\Phi(1) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

$$\Phi(\mathbf{i}) = \begin{pmatrix} -\mathbf{i} & \\ & \mathbf{i} \end{pmatrix}$$

$$\Phi(\mathbf{j}) = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$\Phi(\mathbf{k}) = \begin{pmatrix} & -\mathbf{i} \\ -\mathbf{i} & \end{pmatrix}$$

b: By multilinearity again, it suffices to check the ideal property on generators only. That is, we show:

$$\Phi(x)\Phi(y) - \Phi(y)\Phi(x) \in \mathrm{span}_{\mathbb{R}}\{\Phi(\mathbf{i}), \Phi(\mathbf{j}), \Phi(\mathbf{k})\} =: V$$

for all $x, y \in \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ with $y \neq 1$. Denoting by $[\,\cdot, \cdot]$ the commutator on $\mathrm{Mat}_2(\mathbb{C})$, we calculate:

$$\begin{aligned} [\Phi(1), \Phi(\mathbf{i})] =& [\Phi(1), \Phi(\mathbf{j})] = [\Phi(1), \Phi(\mathbf{k})] \\ =& [\Phi(\mathbf{i}), \Phi(\mathbf{i})] = [\Phi(\mathbf{j}), \Phi(\mathbf{j})] \\ =& [\Phi(\mathbf{k}), \Phi(\mathbf{k})] = 0 \\ [\Phi(\mathbf{i}), \Phi(\mathbf{j})] =& 2\Phi(\mathbf{k}) \\ [\Phi(\mathbf{i}), \Phi(\mathbf{k})] =& -2\Phi(\mathbf{j}) \\ [\Phi(\mathbf{j}), \Phi(\mathbf{k})] =& 2\Phi(\mathbf{i}) \end{aligned}$$

This proves that V is an ideal and also shows $(V, [\,\cdot, \cdot]) \cong (\mathbb{R}^3, 2\otimes)$ by linear extension of:

$$\begin{aligned} \Phi(\mathbf{i}) \mapsto& e_1 \\ \Phi(\mathbf{j}) \mapsto& e_2 \\ \Phi(\mathbf{k}) \mapsto& e_3 \end{aligned}$$

In order to complete the exercise, we show that any Lie algebra $(E, [\,\cdot, \cdot]_E)$ over a field $\mathbb{K}$ is isomorphic to $(E, c[\,\cdot, \cdot]_E)$ for $c \in \mathbb{K}^\times$. Define $\Psi : E \to E$ by $\Psi v := c^{-1} v$. Using bilinearity, we calculate:

$$c[\Psi v, \Psi w]_E = c^{-1}[v, w]_E = \Psi[v, w]_E$$

so that $\Psi : (E, [\,\cdot, \cdot]_E) \to (E, c[\,\cdot, \cdot]_E)$ preserves the Lie bracket. It is clear that Ψ is an isomorphism of vector spaces and thus Ψ becomes an isomorphism of Lie algebras.

References

1. Alexander S. Kechris, *Classical Descriptive Set Theory*, Springer, 1995.
2. Emmanuel Kowalski, *Measure and Integral*, Online Notes, July 2012, `http://www.math.ethz.ch/~kowalski/measure-integral.pdf` (last access August 31, 2014).
3. Leopoldo Nachbin, *The Haar Integral*, D. van Nostrand Company, Inc., 1965.