SOLUTIONS TO SELECTED EXERCISES: SHEET 3

Exercise 1

a: Consider the map $\Psi : \mathrm{GL}_n^+(\mathbb{R}) \to (0,\infty) \times \mathrm{SL}_n(\mathbb{R})$ given by:

$$\Psi(g) := \big((\det g)^{\frac{1}{n}}, (\det g)^{-\frac{1}{n}} g\big)$$

Ψ is continuous and hence $\Psi^{-1} \subseteq (0,1] \times B \subseteq \mathrm{GL}_n^+(\mathbb{R})$ is measureable whenever $B \in \mathcal{B}\,\mathrm{SL}_n(\mathbb{R})$. As $\mathrm{GL}_n^+(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ is open, $\mathcal{C}(B) = \{0\} \cup \Psi^{-1} \subseteq (0,1] \times B$ is measureable in $\mathbb{R}^{n^2}$.

Left invariance of the measure:

$$m_{\mathrm{SL}_n(\mathbb{R})}(B) := m_{\mathbb{R}^{n^2}}\big(\mathcal{C}(B)\big)$$

follows immediately from $\mathcal{C}(gB) = g\mathcal{C}(B)$ and the fact that on $\mathbb{R}^{n^2}$ an element $g \in \mathrm{SL}_n(\mathbb{R})$ acts – after an appropriate permutation of the standard basis – like a block-diagonal $n^2 \times n^2$-matrix with n copies of g along the diagonal.

b: We use the coordinate system $\phi : \mathrm{Aff}_1(\mathbb{R}) \ni (a,b) \mapsto (a, a^{-1}b) \in \mathbb{R}^\times \times \mathbb{R}$. On $\mathbb{R}^\times \times \mathbb{R}$ we define the measure $\mathrm{d}\nu(\alpha,\beta) := \frac{1}{|\alpha|}\,\mathrm{d}\alpha\,\mathrm{d}\beta$ and we claim that $(\phi^{-1})_*\nu$ is a left-Haar measure on $\mathrm{Aff}_1(\mathbb{R})$.

Let $f \in \mathscr{C}_c\big(\mathrm{Aff}_1(\mathbb{R})\big)$ and denote $\psi(\alpha) := x\alpha$, then for left-translation – indicated by subscript – follows:

$$\begin{aligned}
(\phi^{-1})_*\nu\big(f_{\left(\begin{smallmatrix} x & y \\ & 1\end{smallmatrix}\right)}\big) &= \int_{\mathbb{R}^\times}\left(\int_{\mathbb{R}} \frac{f_{\left(\begin{smallmatrix} x & y \\ & 1\end{smallmatrix}\right)} \circ \phi^{-1}(\alpha,\beta)}{|\alpha|}\,\mathrm{d}\beta\right)\mathrm{d}\alpha \\
&= \int_{\mathbb{R}^\times}\left(\int_{\mathbb{R}} \frac{f \circ \phi^{-1}\big(x\alpha, \beta + (x\alpha)^{-1}y\big)}{|\alpha|}\,\mathrm{d}\beta\right)\mathrm{d}\alpha \\
\text{(trans. inv.)} \quad &= \int_{\mathbb{R}^\times}\left(\int_{\mathbb{R}} \frac{f \circ \phi^{-1}(x\alpha, \beta)}{|\alpha|}\,\mathrm{d}\beta\right)\mathrm{d}\alpha \\
(\psi' = x) \quad &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}^\times} \frac{f \circ \phi^{-1}\big(\psi(\alpha), \beta\big)}{|\psi(\alpha)|}\,|\psi'(\alpha)|\,\mathrm{d}\alpha\right)\mathrm{d}\beta \\
(\psi(\mathbb{R}^\times) = \mathbb{R}^\times) \quad &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}^\times} \frac{f \circ \phi^{-1}(\alpha, \beta)}{|\alpha|}\,\mathrm{d}\alpha\right)\mathrm{d}\beta = (\phi^{-1})_*\nu(f)
\end{aligned}$$

and thus we have indeed found a left Haar measure. For right translation – indicated by superscript – follows:

$$\begin{aligned}
(\phi^{-1})_*\nu\big(f^{\left(\begin{smallmatrix} x & y \\ & 1\end{smallmatrix}\right)}\big) &= \int_{\mathbb{R}^\times}\left(\int_{\mathbb{R}} \frac{f^{\left(\begin{smallmatrix} x & y \\ & 1\end{smallmatrix}\right)} \circ \phi^{-1}(\alpha,\beta)}{|\alpha|}\,\mathrm{d}\beta\right)\mathrm{d}\alpha \\
&= \int_{\mathbb{R}^\times}\left(\int_{\mathbb{R}} \frac{f \circ \phi^{-1}\big(x\alpha, x^{-1}\beta + x^{-1}y\big)}{|\alpha|}\,\mathrm{d}\beta\right)\mathrm{d}\alpha \\
\text{(trans. inv.)} \quad &= \int_{\mathbb{R}^\times}\left(\int_{\mathbb{R}} \frac{f \circ \phi^{-1}\big(x\alpha, x^{-1}\beta\big)}{|\alpha|}\,\mathrm{d}\beta\right)\mathrm{d}\alpha
\end{aligned}$$

$$\begin{aligned}
\text{(subst. } \beta \mapsto x\beta) \qquad &= \int_{\mathbb{R}^\times} |x| \left(\int_{\mathbb{R}} \frac{f \circ \phi^{-1}\big(x\alpha, \beta\big)}{|\alpha|} \, \mathrm{d}\beta \right) \mathrm{d}\alpha \\
\text{(as above)} \qquad &= |x| \, (\phi^{-1})_* \nu(f)
\end{aligned}$$

Hence follows $\Delta_{\mathrm{Aff}_1(\mathbb{R})} \left(\begin{smallmatrix} x & y \\ & 1 \end{smallmatrix} \right) = |x|^{-1}$.

Exercise 5

First look at $\rho(\mathrm{Ad}_g^{-1} v) = \det \big(\mathrm{d}_0 \phi_{\mathrm{Ad}_{g^{-1}} v} \big)$. Fix some $\delta < \delta_0$ such that:

$$U := B_\delta \cup \mathrm{Ad}_{g^{-1}} B_\delta \cup \mathrm{Ad}_g B_\delta \subseteq B_{\delta_0}$$

Note that for $v, w \in U$:

$$\phi_{\mathrm{Ad}_{g^{-1}} v}(w) = (\mathrm{Ad}_{g^{-1}} \circ \phi_v \circ \mathrm{Ad}_g)(w)$$

In particular:

$$\rho(\mathrm{Ad}_g^{-1} v) = \det(\mathrm{d}_v \mathrm{Ad}_{g^{-1}})^{-1} \det(\mathrm{d}_0 \mathrm{Ad}_g)^{-1} \rho(v)$$

As $\mathrm{Ad}_{g^{-1}}$ is linear, $\det(\mathrm{d}_v \mathrm{Ad}_{g^{-1}}) = \det(\mathrm{d}_0 \mathrm{Ad}_{g^{-1}}) = \det(\mathrm{d}_0 \mathrm{Ad}_g)^{-1}$ and thus:

$$\rho(\mathrm{Ad}_g^{-1} v) = \rho(v)$$

We calculate:

$$\begin{aligned}
\mu(Bg) =& \mu(g^{-1}Bg) = \sum_k \int_{B_\delta} \mathbb{1}_{g^{-1}Bg \cap P_k} \big(g_k \exp(v) \big) \, \mathrm{d}v \\
=& \sum_k \int_{B_\delta} \mathbb{1}_{B \cap P'_k} \big(g'_k \exp(\mathrm{Ad}_g v) \big) \rho(v) \, \mathrm{d}v \\
=& \frac{1}{|\det \mathrm{d}\mathrm{Ad}_g|} \sum_k \int_{\mathrm{Ad}_g B_\delta} \mathbb{1}_{B \cap P'_k} \big(g'_k \exp(v) \big) \rho(\mathrm{Ad}_{g^{-1}} v) \, \mathrm{d}v \\
=& \big| \det \mathrm{d}\mathrm{Ad}_{g^{-1}} \big| \sum_k \int_{\mathrm{Ad}_g B_\delta} \mathbb{1}_{B \cap P'_k} \big(g'_k \exp(v) \big) \rho(v) \, \mathrm{d}v \\
=& \big| \det \mathrm{d}\mathrm{Ad}_{g^{-1}} \big| \, \mu(B)
\end{aligned}$$

where we used that for a fixed measure on $\mathfrak{g}$, the Haar measure constructed in the above manner does not depend on the choice of the neighbourhood $U = B_\delta$ of 0. As this is by itself an interesting fact, we give the construction of the Haar measure for neighbourhoods of 0 which are not necessarily open balls around 0.

Let $V \subseteq \mathfrak{g}$ an open neighbourhood of 0 such that $\exp(V) \subseteq G$ is open and $\exp\big|_V$ is a diffeomorphism onto its image. Denote $U := \exp(V)$. As exp and $p : G \times G \to G$ are continuous, there is some open neighbourhood $A \subseteq \mathfrak{g}$ of 0 such that $\exp(A)\exp(A) \subseteq U$ and hence we obtain a well-defined product on $A \times A$:

$$\begin{aligned}
* :& A \times A \to \mathfrak{g} \\
v * w :=& \log \big(\exp(v) \exp(w) \big)
\end{aligned}$$

By the same argument, we can assume that A is chosen so that the above holds and $\exp(u)\exp(v)\exp(w) \in U$ for all $u, v, w \in A$. We note that $\det(\mathrm{d}_0 \phi_0) = 1$ as $\phi_0 = \mathrm{id}$. As $v \mapsto \det(\mathrm{d}_0 \phi_v)$ is continuous, we can assume without loss of generality that A is chosen so that for $v, w \in A$ holds $\det(\mathrm{d}_v \phi_w) > 0$.

Lemma 0.1. *Let $v_0 \in A$, $f \in \mathscr{C}_c(G)$, then:*

$$\int_{*A} f\big(\exp(v_0) \exp(v) \big) \rho(v) \, \mathrm{d}v = \int_{v_0 * A} f\big(\exp(v) \big) \rho(v) \, \mathrm{d}v$$

Proof. If $v_0, v, w \in A$, then $\phi_{\phi_{v_0} v}(w) = (\phi_{v_0} v) * w = (v_0 * v) * w = v_0 * (v * w) = \phi_{v_0} \circ \phi_v(w)$, hence $\mathrm{d}_0 \phi_{\phi_{v_0} v} = \mathrm{d}_v \phi_{v_0} \, \mathrm{d}_0 \phi_v$ and thus $\rho(\phi_{v_0} v) = \det(\mathrm{d}_v \phi_{v_0}) \rho(v)$, yielding:

$$\begin{aligned}\int_A f\big(\exp(v_0)\exp(v)\big)\rho(v)\,\mathrm{d}v &= \int_A f\big(\exp(\phi_{v_0} v)\big)\rho(\phi_{v_0} v)\det(\mathrm{d}_v \phi_{v_0})\,\mathrm{d}v \\ &= \int_{v_0 * A} f\big(\exp(v)\big)\rho(v)\,\mathrm{d}v\end{aligned}$$

□

Given this local definition of the measure, we define:

$$m(B) := \sum_{k\geq 1} \int_A \mathbb{1}_{B\cap P_k}\big(g_k \exp(v)\big)\rho(v)\,\mathrm{d}v \quad \forall B \in \mathcal{B}G$$

where we chose a symmetric neighbourhood $O \subseteq \mathfrak{g}$ of 0 such that $O * O * O \subseteq A$, $(g_k)_{k\in\mathbb{N}} \in G^{\mathbb{N}}$ and $(P_k)_{k\in\mathbb{N}} \in \mathcal{B}G^{\mathbb{N}}$ such that:

$$\begin{aligned} G = \bigsqcup_{k\geq 1} P_k &= \bigcup_{k\geq 1} g_k \exp(O) \\ P_k \subseteq &g_k \exp(O) \quad \forall k \geq 1\end{aligned}$$

We check that this is independent of the choice of the partition and the sequence. Assume that $(Q_l)_{l\in\mathbb{N}} \in \mathcal{B}G^{\mathbb{N}}$ and $(h_l)_{l\in\mathbb{N}} \in G^{\mathbb{N}}$ are chosen similarly. Assume that $P_k \cap Q_l \neq \emptyset$, then there are $v_k, w_l \in O$ such that:

$$g_k \exp(v_k) = h_l \exp(v_l) \Rightarrow h_l^{-1} g_k \in \exp(O * O)$$

Beppo-Levi implies:

$$\begin{aligned} m(B) &= \sum_{k\geq 1} \int_A \mathbb{1}_{B\cap P_k}\big(g_k \exp(v)\big)\rho(v)\,\mathrm{d}v \\ &= \sum_{k\geq 1} \int_O \mathbb{1}_{B\cap P_k}\big(g_k \exp(v)\big)\rho(v)\,\mathrm{d}v \\ &= \sum_{k,l\geq 1} \int_O \mathbb{1}_{B\cap P_k\cap Q_l}\big(h_l \exp(v_{k,l})\exp(v)\big)\rho(v)\,\mathrm{d}v \\ &= \sum_{k,l\geq 1} \int_{v_{k,l} * O} \mathbb{1}_{B\cap P_k\cap Q_l}\big(h_l \exp(v)\big)\rho(v)\,\mathrm{d}v \\ &= \sum_{k,l\geq 1} \int_A \mathbb{1}_{B\cap P_k\cap Q_l}\big(h_l \exp(v)\big)\rho(v)\,\mathrm{d}v \\ &= \sum_{l\geq 1} \int_A \mathbb{1}_{B\cap Q_l}\big(h_l \exp(v)\big)\rho(v)\,\mathrm{d}v \end{aligned}$$

where in the second to last step we used that for $v \in A$ holds $h_l \exp(v) \in P_k$ iff $v = v_{k,l} * w$ for some $w \in O$, which can be checked easily using the definitions of A and O. Left-invariance of this measure follows by the same argument as in class: if $(P_k)_{k\in\mathbb{N}}$ is a partition corresponding to $O \subseteq A$ and the sequence $(g_k)_{k\in\mathbb{N}}$, then $(g^{-1}P_k)_{k\in\mathbb{N}}$ is a partition corresponding to $O \subseteq A$ and the sequence $(g^{-1}g_k)_{k\in\mathbb{N}}$. Finally we note that the measure is independent of the specific choice of O and A for some fixed choice of measure on $\mathfrak{g}$. In order to see this, assume that A_1, O_1 and A_2, O_2 are two distinct choices as above. Let $\exp(O_1) \cap \exp(O_2) =: U$. The measures defined by (A_1, O_1) and (A_2, O_2) agree on U without vanishing which follows from the fact that we can choose U as an element of both partitions defining the measures.

References

1. Alexander S. Kechris, *Classical Descriptive Set Theory*, Springer, 1995.