SOLUTIONS TO SELECTED EXERCISES: SHEET 4

Exercise 2

(a): First assume that w is an eigenvector of H, then:

$$HXw = XHw + [H, X]w = \lambda Xw + 2Xw = (\lambda + 2)Xw$$

so Xw is an eigenvector of H for eigenvalue $\lambda+2$. Maximality of $\Re\lambda$ implies $Xw = 0$. Now assume that $X(H-\lambda)^m w = 0$ for some $m \geq 1$, then:

$$HX(H-\lambda)^{m-1}w = (\lambda+2)X(H-\lambda)^{m-1}w$$

implying that $X(H-\lambda)^{m-1}w = 0$ by maximality. This proves the general case.

(b): We claim that Yw is a generalized eigenvector for the eigenvalue $\lambda - 2$ and if w is an eigenvector, so is Yw. Note that $H^r Y = Y(H-2)^r$ for all $r \geq 0$. Hence the binomial formula implies:

$$\big(H - (\lambda-2)\big)^m Y = Y(H-\lambda)^m$$

If we choose $m \in \mathbb{N}$ such that $(H-\lambda)^m w$ is an eigenvector for eigenvalue λ of H, then this yields

$$H\big(H-(\lambda-2)\big)^m Yw = (\lambda-2)\big(H-(\lambda-2)\big)^m Y_w$$

as desired. As V is finite dimensional, H has only finitely many distinct eigenvalues and thus there exists N such that $Y^N w = 0$.

(c): Let $W := \operatorname{span}_{\mathbb{C}}\{w_0, \ldots, w_n\}$. It is clear that $Yw_i \in W$ by assumption on n and thus linearity implies $YW \subseteq W$. Part (b) shows that w_i is an eigenvector of H for every $0 \leq i \leq n$, hence $HW \subseteq W$. We know that $Xw_0 \in W$. Assume that $XY^k w \in W$, then:

$$XY^{k+1}w = \underbrace{YXY^k w}_{\in W} + \underbrace{HY^k w}_{\in W} \in W$$

hence $XW \subseteq W$ follows inductively. Thus W is $\mathfrak{sl}_2(\mathbb{C})$-invariant and irreducibility yields $W = V$. The w_i are linearly independent, as they are eigenvectors for H for distinct eigenvalues. It follows easily by induction that $XY^k w = k(\lambda-k+1)Y^{k-1}w$, where $0Y^{-1}w := 0$, hence for $r_k := k(\lambda-k+1)$ holds:

$$H \equiv \begin{pmatrix} \lambda & & \\ & \lambda-2 & \\ & & \ddots \end{pmatrix}$$

$$Y \equiv \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \ddots \end{pmatrix}$$

$$X \equiv \begin{pmatrix} 0 & & & \\ r_1 & 0 & & \\ & r_2 & 0 & \ddots \end{pmatrix}$$

(d): Let n as in part (c), then the above calculation shows:

$$0 = (n+1)(\lambda - n)Y^n w$$

and as $Y^n w \neq 0$, it follows $\lambda = n$. In particular, the eigenvalues of H are given by $\{n, n-2, \dots, -n+2, -n\}$. The uniqueness of the irreducible representation of dimension $n+1$ follows, as it is completely determined by H (as we have proven in parts (a) to (c)).

Exercise 3

Let $G \subseteq \mathrm{GL}_n(\mathbb{R})$ a closed subgroup with Lie algebra $\mathfrak{g}$, then we can find an open neighborhood of the origin $V \subseteq \mathfrak{g}$ such that $\exp(V) \subseteq G$ is an open neighbourhood of the identity and $\exp\big|_V$ is a diffeomorphism onto its image, whose inverse we denote log. Let $U \subseteq V$ an open neighbourhood of the origin such that $\exp(U)\exp(U) \subseteq \exp(V)$ and denote:

$$X * Y := \log\big(\exp(X)\exp(Y)\big) \quad \forall X, Y \in U$$

This is a smooth product structure on U. Recall the Baker-Campbell-Hausdorff formula:

Theorem 0.1. *There is a sequence of homogeneous Lie polynomials $Z_n \in \mathbb{R}[x, y]$ of degree $\deg Z_n = n$ such that:*

$$X * Y = \sum_{n \geq 1} Z_n(X, Y) \quad \forall X, Y \in U$$

If $\mathfrak{g}$ is nilpotent, then there is $N \in \mathbb{N}$ such that $Z_n = 0$ for all $n > N$. The map:

$$* : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (X, Y) \mapsto X * Y := \sum_{n=1}^{N} Z_n(X, Y)$$

is clearly smooth and extends $*$ as defined on $U \times U$. We need to show that $\mathfrak{g}$ becomes a group under this definition of the product. First of all $0 \in \mathfrak{g}$ indeed becomes an identity element, as the inductive formula for Z_n obtained in class:

$$\begin{aligned} Z_1 =& X + Y \\ (n+1)Z_{n+1} =& \frac{1}{2}[X - Y, Z_n] \\ &+ \sum_{\substack{p \geq 1 \\ 2p \leq n}} k_{2p} \sum_{\substack{r_i \geq 1 \\ r_1 + \dots + r_{2p} = n}} [Z_{r_1}, [Z_{r_2}, \dots [Z_{r2p}, X + Y] \dots]] \end{aligned}$$

If $X = 0$ or $Y = 0$, then the above formulae yield $Z_2 = 0$ immediately. Assume that $n \geq 2$ and $Z_2, \dots, Z_n = 0$, then the above formula yields $(n+1)Z_{n+1} = 0$. Let $n \geq r_1, \dots, r_k \geq 1$, then:

$$[Z_{r_1}, \dots [Z_{r_k}, X + Y] \dots] = 0$$

If $r_i \neq 1$ for some i, this follows from the assumption. Otherwise $Z_{r_k} = Z_1 = X + Y$ and hence $[X + Y, X + Y] = 0$ implies that the above expression vanishes. In particular:

$$\sum_{\substack{r_i \geq 1 \\ r_1 + \dots + r_{2p} = n}} [Z_{r_1}, [Z_{r_2}, \dots [Z_{r_{2p}}, X + Y] \dots]] = 0 \quad \forall p \geq 1 : 2p \leq n$$

and thus $0 * X = X * 0 = X$ for all $X \in \mathfrak{g}$. For existence of inversion we note that $Z_1(-X, X) = 0$ and hence induction again implies that $Z_n(-X, X) = 0$ for all $n \in \mathbb{N}$. Hence $(\mathfrak{g}, *)$ is a group and the group operations are obviously smooth. On a neighbourhood of $0 \in \mathfrak{g}$, the exponential map $X \mapsto \exp(X)$ is a diffeomorphism, hence $\mathrm{d}_0\exp$ is an isomorphism of vector spaces $T_0\mathfrak{g}$ and $T_1 G$. Furthermore we note that for $X, Y \in U$ holds $\exp(X * Y) = \exp(X)\exp(Y)$, i.e. on a neighbourhood of $0 \in \mathfrak{g}$, exp is an isomorphism of groups. We claim that hence the Lie algebra of G and of $\mathfrak{g}$ agree. More generally we prove the following generalization of the result shown in class:

Proposition 0.2. *Let G, H Lie groups and $\phi : G \to H$ smooth such that on a neighbourhood U of $1 \in G$ holds:*

$$\phi(gh) = \phi(g)\phi(h) \quad \forall g, h \in U$$

Then $\mathrm{d}_1\phi$ is a homomorphism of Lie algebras.

Proof. Let $v \in T_1G$ and $w := \mathrm{d}\phi v$. Denote by $V \in \mathfrak{g}$ and $W \in \mathfrak{h}$ the corresponding left invariant vector fields. Let $f \in \mathscr{C}^\infty(H)$, then:

$$(l_g^* \circ \phi^*)(f) = (\phi^* \circ l_{\phi(g)}^*)(f) \quad \forall g \in U$$

on U by assumption. Hence:

$$W_{\phi(g)}(f) = w(f \circ l_{\phi(g)}) = v(f \circ \phi \circ l_g) = V_g(f \circ \phi) \quad \forall g \in U$$

As f was arbitrary, $W \circ \phi = \mathrm{d}\phi V$ on U, i.e. as vector fields on U they are ϕ-related. Hence if $r \in T_1G$, $s := \mathrm{d}\phi r \in T_1H$ and R, S the corresponding left-invariant vector fields, then:

$$[\mathrm{d}_1\phi v, \mathrm{d}_1\phi r] = [W, S] \circ \phi(1) = (\mathrm{d}\phi[V, R])_1 = \mathrm{d}_1\phi[V, R]_1 = \mathrm{d}_1\phi[v, r]$$

by definition. □

Now we can apply the cited theorem to deduce that $G \cong \mathfrak{g}$ where the latter is equipped with the $*$-operation.