# SELECTED SOLUTIONS TO EXERCISE SHEET 5

MANUEL W. LUETHI

## Exercise 2

**(b):** In what follows, assume that $R$ is a ring and $\mathfrak{g}$ is a Lie algebra over $R$, i.e. $\mathfrak{g}$ is an $R$-module equipped with a $R$-bilinear, anticommutative map $[\cdot,\cdot]$ satisfying the Jacobi identity. Given a commutative $R$-algebra $A$, we define the Lie algebra tensored by $A$ over $R$ as the $R$-module $\mathfrak{g}\otimes_R A$ equipped with the Lie bracket defined on simple tensors as:

$$[v\otimes a, w\otimes b] := [v,w]\otimes ab \quad \forall v,w\in\mathfrak{g}\forall a,b\in A$$

Recall that for any two $R$-algebras $A$ and $B$ the tensor product $A\otimes_R B$ becomes an $R$-algebra by $(a\otimes b)(\alpha\otimes\beta) := a\alpha\otimes b\beta$. Finally recall that for a ring $R$ and $R$-modules $M,N,P$ the following are true:

$$\begin{aligned} M\otimes_R(N\otimes_R P) &\cong (M\otimes_R N)\otimes_R P \\ M\otimes_R(N\oplus P) &\cong (M\otimes_R N)\oplus(M\otimes_R P) \\ M\otimes_R R &\cong M \end{aligned}$$

Over $\mathbb{R}$ we observe the following chain of isomorphisms:

$$\begin{aligned} \mathfrak{sl}_2(\mathbb{C})\otimes_\mathbb{R}\mathbb{C} &\overset{(1)}{\cong} (\mathfrak{sl}_2(\mathbb{R})\otimes_\mathbb{R}\mathbb{C})\otimes_\mathbb{R}\mathbb{C} \\ &\overset{(2)}{\cong} \mathfrak{sl}_2(\mathbb{R})\otimes_\mathbb{R}(\mathbb{C}\otimes_\mathbb{R}\mathbb{C}) \\ &\overset{(3)}{\cong} \mathfrak{sl}_2(\mathbb{R})\otimes_\mathbb{R}(\mathbb{C}\oplus\mathbb{C}) \\ &\overset{(4)}{\cong} \big(\mathfrak{sl}_2(\mathbb{R})\otimes_\mathbb{R}\mathbb{C}\big)\oplus\big(\mathfrak{sl}_2(\mathbb{R})\otimes_\mathbb{R}\mathbb{C}\big) \\ &\overset{(5)}{\cong} \mathfrak{sl}_2(\mathbb{C})\oplus\mathfrak{sl}_2(\mathbb{C}) \end{aligned}$$

The above isomorphisms are immediate as isomorphisms of $\mathbb{R}$ modules, using for example $\mathbb{C}\otimes_\mathbb{R}\mathbb{C}\cong(\mathbb{R}\oplus\mathbb{R})\otimes\mathbb{C}\cong\mathbb{C}\oplus\mathbb{C}$ and the isomorphism $\mathfrak{sl}_2(\mathbb{C})\cong\mathfrak{sl}_2(\mathbb{R})\otimes_\mathbb{R}\mathbb{C}$ discussed extensively already. For the natural isomorphisms (2) and (4) as well as for the well-known isomorphism $\mathfrak{sl}_2(\mathbb{C})\cong\mathfrak{sl}_2(\mathbb{R})\otimes_\mathbb{R}\mathbb{C}$ involved in (1) and (5) one easily checks directly that everything is defined over $\mathbb{C}$ and the isomorphisms are homomorphisms of Lie algebras. However, for the obvious isomorphism $\mathbb{C}\otimes_\mathbb{R}\mathbb{C}\cong\mathbb{C}\oplus\mathbb{C}$, this does not work: the above argument for $\mathbb{C}\cong\mathbb{R}\otimes\mathbb{R}$ only considers $\mathbb{C}$ as an $\mathbb{R}$ module and omits the algebra structure on $\mathbb{C}$. Hence we need to show that there exists an isomorphism $\mathbb{C}\otimes_\mathbb{R}\mathbb{C}\cong\mathbb{C}\oplus\mathbb{C}$ of $\mathbb{R}$-algebras. Consider the element $\omega := \mathbf{i}\otimes\mathbf{i}$, then:

$$\begin{aligned} \omega^2 &= \frac{1}{4}1\otimes 1 - \frac{1}{2}(1\otimes 1)(\mathbf{i}\otimes\mathbf{i}) + \frac{1}{4}(\mathbf{i}\otimes\mathbf{i})(\mathbf{i}\otimes\mathbf{i}) \\ &= \frac{1}{2}1\otimes 1 - \frac{1}{2}\mathbf{i}\otimes\mathbf{i} = \omega \end{aligned}$$

*Date*: December 8, 2015.

This property (*idempotence*) is of use in much more generality:

**Proposition 0.1.** *Let $S, R$ unital, commutative rings, $R \to S$ a homomorphism and $\omega \in S \setminus \{0,1\}$ an idempotent, i.e. $\omega^2 = \omega$. Then there are unital $R$ algebras $A$ and $B$ such that:*

$$S \cong A \oplus B$$

*as $R$-algebras.*

*Proof.* Let $\overline{\omega} := 1 - \omega$, then $\overline{\omega}^2 = 1 - 2\omega + \omega^2 = 1 - \omega$. $\overline{\omega} \in S \setminus \{0,1\}$ by assumptions on $\omega$. The principal ideals $(\omega)$, $(\overline{\omega}) \subseteq S$ are $R$-subalgebras by idempotence of $\omega$, $\overline{\omega}$. Let $s \in S$, then $s\omega + s\overline{\omega} = s$, thus $S = (\omega) + (\overline{\omega})$. It remains to show that $(\omega) \cap (\overline{\omega}) = \{0\}$. Assume $s_1, s_2 \in S$ such that $s_1\omega = s_2\overline{\omega}$, then $s_2 = (s_1 + s_2)\omega$ and in particular $s_1\omega = 0$. Hence follows $(\omega) \cap (\overline{\omega}) = \{0\}$. It is clear that the ring isomorphism $S \cong (\omega) \oplus (\overline{\omega})$, $s \mapsto (s\omega, s\overline{\omega})$ is $R$-linear. Note that $\omega x = x$ and $\overline{\omega} y = y$ for all $x \in (\omega)$, $y \in (\overline{\omega})$. □

**Corollary 0.2.** *Let $\mathfrak{g}$ be a Lie algebra over $R$ and $S$ a unital, commutative $R$-algebra with an idempotent $\omega \in S \setminus \{0,1\}$, let $\Phi : S \to (\omega) \oplus (\overline{\omega})$ the isomorphism from above. Then $\mathrm{id} \otimes \Phi$ defined on simple tensors by:*

$$\mathrm{id} \otimes \Phi(v \otimes s) := v \otimes \Phi(s) \quad \forall v \in \mathfrak{g} \forall s \in S$$

*is an isomorphism of Lie algebras.*

*Proof.* The fact that $\mathrm{id} \otimes \Phi$ is a Lie algebra homomorphism is obvious from the definition of the bracket on $\mathfrak{g} \otimes S$ and $\mathfrak{g} \otimes \big((\omega) \oplus \overline{\omega}\big)$. Clearly $\mathrm{id} \otimes \Phi$ is invertible, hence an isomorphism. □

In the special case $S = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\omega = \frac{1}{2} 1 \otimes 1 - \frac{1}{2} \mathbf{i} \otimes \mathbf{i}$, we note that the isomorphism – call it $\psi$ – is clearly defined over $\mathbb{C}$: the complex structure on $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is given by the homomorphism $z \mapsto i(z) := 1 \otimes z$. We want to show that $(\omega) \cong \mathbb{C}$ and $\overline{\omega} \cong \mathbb{C}$ as $\mathbb{R}$-algebras and over $\mathbb{C}$ in order to obtain isomorphism (3). Define maps $\mathbb{C} \to (\omega)$ and $\mathbb{C} \to (\overline{\omega})$ by $z \mapsto i(z)\omega$ and $z \mapsto i(z)\overline{\omega}$. As $\omega$ and $\overline{\omega}$ are idempotents, these maps are homomorphism of unital rings, and as $\mathbb{C}$ is a field, they are injective. In particular, the image of $\mathbb{C}$ has $\mathbb{R}$-dimension 2 both in $(\omega)$ and in $(\overline{\omega})$. As $\mathbb{C} \otimes \mathbb{C}$ has $\mathbb{R}$-dimension 4, it follows that $\mathbb{C} \cong (\omega)$ and $\mathbb{C} \cong (\overline{\omega})$. Let $\imath_\omega : (\omega) \to \mathbb{C}$ and $\imath_{\overline{\omega}} : (\overline{\omega}) \to \mathbb{C}$ denote the inverses of these isomorphisms. These maps being inverses of ring isomorphisms, we note that $\forall z \in \mathbb{C}, x \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$:

$$\imath_\omega\big(i(z)x\omega\big) = \imath_\omega\big(i(z)\omega\big)\imath_\omega\big(x\omega\big) = z\imath_\omega\big(x\omega\big)$$

Hence the isomorphism $(\imath_\omega, \imath_{\overline{\omega}}) \circ \psi : \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$ is an isomorphism of $\mathbb{C}$-algebras:

$$\begin{aligned} i(z)x \mapsto & \big(i(z)x\omega, i(z)x\overline{\omega}\big) \mapsto \big(z\imath_\omega(x\omega), z\imath_{\overline{\omega}}(x\overline{\omega})\big) \\ =& z\big(\imath_\omega^{-1}(x\omega), \imath_{\overline{\omega}}^{-1}(x\overline{\omega})\big) = z\big((\imath_\omega, \imath_{\overline{\omega}}) \circ \psi\big)\big(x\big) \end{aligned}$$

## Exercise 3

Assume that $\Phi : G \to G$ is an automorphism, then $\mathrm{d}_1\Phi : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of the Lie algebra $\mathfrak{g}$.

**(a):** We need to show that every automorphism of $\mathfrak{sl}_2(\mathbb{C})$ is given by $\mathrm{Ad}_g$ for some $g \in \mathrm{SL}_2(\mathbb{C})$. As $\mathrm{SL}_2(\mathbb{C})$ is simply connected, the automorphism $\Phi : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C})$ satisfying $\mathrm{d}_1\Phi = \mathrm{Ad}_g$ is unique and thus follows $\Phi = c_g$. Consider the standard representation of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathbb{C}^2$ and note that this representation os irreducible: if $V \subseteq \mathbb{C}^2$ is a proper invariant subspace, then on $V$ the representation is trivial, i.e. $V = \mathbb{C}v$ for some non-zero common eigenvector $v$ of $\mathfrak{sl}_2(\mathbb{C})$. The eigenvectors of $h$ are known and well-known not to be eigenvectors of $e, f$ yielding a contradiction. The definition:

$$(a, v) \mapsto \psi(a)v \quad \forall a \in \mathfrak{sl}_2(\mathbb{C}) \forall v \in \mathbb{C}^2$$

yields another representation of $\mathfrak{sl}_2(\mathbb{C})$, which is irreducible by the same argument as above. In particular, there is an isomorphism $\Psi : \mathbb{C}^2 \to \mathbb{C}^2$ such that:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{a} & \mathbb{C}^2 \\ \Psi \downarrow & & \downarrow \Psi \\ \mathbb{C}^2 & \xrightarrow[\psi(a)]{} & \mathbb{C}^2 \end{array} \quad \forall a \in \mathfrak{sl}_2(\mathbb{C})$$

In particular, $\psi(h)$ has eigenvalues $\{\pm 1\}$, thus $\psi(h) = \operatorname{Ad}_g h$ for some $g \in \mathrm{SL}_2(\mathbb{C})$. An explicit calculation of $[h, a]$ for general $a \in \mathfrak{sl}_2(\mathbb{C})$ together with the automorphism property of $\psi$ show that there is $\lambda \in \mathbb{C}^\times$:

$$\psi(e) = \lambda \operatorname{Ad}_g e \quad \text{and} \quad \psi(f) = \lambda^{-1} \operatorname{Ad}_g f$$

But then $a(\lambda) := \begin{pmatrix} \sqrt{\lambda} & \\ & \sqrt{\lambda}^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ (for some choice of the square root) satisfies $\operatorname{Ad}_{a(\lambda)} h = h$, $\operatorname{Ad}_{a(\lambda)} e = \lambda e$, $\operatorname{Ad}_{a(\lambda)} f = \lambda^{-1} f$ and thus $\psi = \operatorname{Ad}_{ga(\lambda)}$.

**(b):** Assume that $\psi \in \operatorname{Aut}\big(\mathfrak{sl}_2(\mathbb{R})\big)$. There is a unique $\psi_{\mathbb{C}} \in \operatorname{Aut}\big(\mathfrak{sl}_2(\mathbb{C})\big)$ such that:

$$\begin{array}{ccc} \mathfrak{sl}_2(\mathbb{R}) & \xrightarrow{\psi} & \mathfrak{sl}_2(\mathbb{R}) \\ i \downarrow & & \downarrow i \\ \mathfrak{sl}_2(\mathbb{C}) & \xrightarrow[\psi_{\mathbb{C}}]{} & \mathfrak{sl}_2(\mathbb{C}) \end{array}$$

where $i : \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{sl}_2(\mathbb{C})$ is the canonical embedding $i(a) := a \otimes 1$. We know that $\psi_{\mathbb{C}} = \operatorname{Ad}_g$ for some $g \in \mathrm{SL}_2(\mathbb{C})$ and $g$ fixes $\mathfrak{sl}_2(\mathbb{R}) \subseteq \mathfrak{sl}_2(\mathbb{C})$. A calculation shows that either $g$ is purely real or purely imaginary. Let $g \in \mathrm{SL}_2(\mathbb{C}) \cap \mathbf{i}\operatorname{Mat}_2(\mathbb{R})$ and $\imath := \begin{pmatrix} \mathbf{i} & \\ & -\mathbf{i} \end{pmatrix}$, then $g\imath^{-1} \in \mathrm{SL}_2(\mathbb{R})$. Note that $\operatorname{Ad}_\imath \mathfrak{sl}_2(\mathbb{R}) \subseteq \mathfrak{sl}_2(\mathbb{R})$. Hence $\operatorname{Aut}\big(\mathfrak{sl}_2(\mathbb{R})\big) \cong \operatorname{Ad}_{\mathrm{SL}_2(\mathbb{R})} \rtimes \mathbb{Z}/2\mathbb{Z}$ where the semi-direct product is defined using:

$$\rho : \mathbb{Z}/2\mathbb{Z} \to \operatorname{Aut}(\operatorname{Ad}_{\mathrm{SL}_2(\mathbb{R})}), \quad \rho_1(\operatorname{Ad}_g) := \operatorname{Ad}_\imath \circ \operatorname{Ad}_g \circ \operatorname{Ad}_\imath^{-1}$$

As the center of $\mathrm{SL}_2(\mathbb{R})$ is $\{\pm 1\}$, we know that $\operatorname{Ad}_{\mathrm{SL}_2(\mathbb{R})}$ is in 1-1 correspondence with $\operatorname{Inn}\big(\mathrm{SL}_2(\mathbb{R})\big)$. Note that $g \mapsto \imath g \imath^{-1}$ defines a non-trivial automorphism of $\mathrm{SL}_2(\mathbb{R})$ with differential $\operatorname{Ad}_\imath$ at 1. As $\mathrm{SL}_2(\mathbb{R})$ is connected, if any two automorphisms on $\mathrm{SL}_2(\mathbb{R})$ have the same differential at 1, they agree. Hence differentiation at the identity yields an isomorphism $\operatorname{Aut}\big(\mathrm{SL}_2(\mathbb{R})\big) \cong \operatorname{Aut}\big(\mathfrak{sl}_2(\mathbb{R})\big)$, thus:

$$\begin{aligned} \operatorname{Out}\big(\mathrm{SL}_2(\mathbb{R})\big) &\cong \operatorname{Aut}\big(\mathrm{SL}_2(\mathbb{R})\big) / \operatorname{Inn}\big(\mathrm{SL}_2(\mathbb{R})\big) \\ &\cong \operatorname{Aut}\big(\mathfrak{sl}_2(\mathbb{R})\big) / \operatorname{Ad}_{\mathrm{SL}_2(\mathbb{R})} \cong \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

*E-mail address*: manuel.luethi@math.ethz.ch