SELECTED SOLUTIONS TO EXERCISE SHEET 6

MANUEL W. LUETHI

Exercise 1

Note that the Cartan-Killing B form on a simple Lie algebra $\mathfrak{g}$ is non-trivial. Assume otherwise, then $\mathfrak{g}$ is solvable as of Cartan's criterion, thus $[\mathfrak{g}, \mathfrak{g}] = \{0\}$ and $\mathfrak{g}$ is abelian, thus $x \in \mathfrak{g} \Rightarrow \mathbb{R}x \lhd \mathfrak{g}$, so that $\dim \mathfrak{g} = 1$. This contradicts the definition of simplicity.

Next we digress on the complexification of a bilinear form:

Proposition 0.1. *Let V be a real vector space, $\sigma : V \times V \to \mathbb{R}$ bilinear, then there exists a unique $\mathbb{C}$-(skew-)bilinear $\sigma_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ such that $\sigma_{\mathbb{C}}\big|_{V\times V} \equiv \sigma$. If σ is non-degenerate, so is $\sigma_{\mathbb{C}}$.*

Proof. Assume that τ is any such extension of σ, then $\mathbb{C}$-bilinearity implies that:

$$\tau(v \otimes \alpha, w \otimes \beta) = \alpha\beta\sigma(v, w) \quad \forall \alpha, \beta \in \mathbb{C}\forall v, w \in V$$

Similarly for skew-bilinearity. This shows both existence and uniqueness of $\sigma_{\mathbb{C}}$. For non-degeneracy, assume that $x = \sum_{i=1}^{N} v_i \otimes \alpha_i \in \mathrm{rad}(\sigma_{\mathbb{C}})$, where $\alpha_i \in \mathbb{C}$ and $\{v_1, \ldots, v_N\}$ is a basis of V. If $w \in V$, then:

$$\begin{aligned} 0 =& \sigma_{\mathbb{C}}(x, w \otimes 1) = \sum_{i=1}^{N} \alpha_i \sigma_{\mathbb{C}}(v_i \otimes 1, w \otimes 1) \\ =& \sum_{i=1}^{N} \Re\alpha_i \sigma(v_i, w) + \mathbf{i} \sum_{i=1}^{N} \Im\alpha_i \sigma(v_i, w) \end{aligned}$$

As w is arbitrary, it follows that $\sum_{i=1}^{N} \Re\alpha_i v_i = \sum_{i=1}^{N} \Im\alpha_i v_i = 0$ and thus $x = 0$. □

If $\sigma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is degenerate, $\{0\} \neq \mathrm{rad}(\sigma) \lhd \mathfrak{g}$ and hence σ is trivial by simplicity, which in particular shows that σ is a multiple of B. Assume that σ is non-degenerate. Denote by $\sigma_{\mathbb{C}}^* : \mathfrak{g} \to \mathfrak{g}^*$ the identification given by $\sigma_{\mathbb{C}}^*(a)b := \sigma_{\mathbb{C}}(a, b)$ for all $a, b \in \mathfrak{g}_{\mathbb{C}}$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $(B_{\mathbb{C}}^*)^{-1} \circ \sigma_{\mathbb{C}}^*$. Let $x \in \mathfrak{g}_{\mathbb{C}}$ be an eigenvector for the eigenvalue λ, i.e. $\big((B_{\mathbb{C}}^*)^{-1} \circ \sigma_{\mathbb{C}}^*\big)(x) = \lambda x$, or equivalently $\sigma_{\mathbb{C}}^*(x) = \lambda B_{\mathbb{C}}^*(v)$, then in particular:

$$\sigma_{\mathbb{C}}(x, y) - \lambda B_{\mathbb{C}}(x, y) = 0 \quad \forall y \in \mathfrak{g}_{\mathbb{C}}$$

If $\mathfrak{g}$ is absolutely simple, we are done: skew-symmetry of $\mathrm{ad}_{\mathfrak{g}_{\mathbb{C}}}$ with respect to $\sigma_{\mathbb{C}} - \lambda B_{\mathbb{C}}$ implies that $\mathrm{rad}(\sigma_{\mathbb{C}} - \lambda B_{\mathbb{C}}) \lhd \mathfrak{g}_{\mathbb{C}}$ and as of non-triviality $\mathrm{rad}(\sigma_{\mathbb{C}} - \lambda B_{\mathbb{C}}) = \mathfrak{g}_{\mathbb{C}}$ and in particular $\mathfrak{g} \subseteq \mathrm{rad}(\sigma_{\mathbb{C}} - \lambda B_{\mathbb{C}})$. Using non-degeneracy of B we can choose $a, b \in \mathfrak{g}$ such that $B(a, b) \neq 0$, hence:

$$\sigma(a, b) = \sigma_{\mathbb{C}}(a \otimes 1, b \otimes 1) = \lambda B_{\mathbb{C}}(a \otimes 1, b \otimes 1) = \lambda B(a, b)$$

and as of $\sigma(a, b) \in \mathbb{R}$, it follows that $\lambda \in \mathbb{R}$. This implies that $\sigma_{\mathbb{C}} - \lambda B_{\mathbb{C}} = (\sigma - \lambda B)_{\mathbb{C}}$ for the $\mathbb{R}$-valued bilinear form $\sigma - \lambda B$ and as of the preceding discussion, $\sigma - \lambda B$ is degenerate. As $\mathrm{ad}_{\mathfrak{g}}$ is skew-symmetric with respect to $\sigma - \lambda B$, it follows that $\sigma = \lambda B$.

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If $\mathfrak{g}$ is not absolutely simple, there is a bit more work to be done towards a result of interest in itself.

Proposition 0.2. *Let $\mathfrak{g}$ be a real, simple Lie algebra that is not absolutely simple. Then $\mathfrak{g}_{\mathbb{C}}$ is semisimple with two simple summand $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{a} \oplus \mathfrak{b}$ and $\mathfrak{a}$, $\mathfrak{b}$ are isomorphic to $\mathfrak{g}$ over $\mathbb{R}$. Moreover, $\mathfrak{g}$ is a complex, simple Lie algebra.*

Proof. Assume that $\mathfrak{g}$ is not absolutely simple, thus there exists a non-trivial ideal $\mathfrak{a} \lhd \mathfrak{g}_{\mathbb{C}}$: $\{0\} \subsetneq \mathfrak{a} \subsetneq \mathfrak{g}_{\mathbb{C}}$. We define a conjugation $\bar{\cdot} : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ defined by extending $\overline{v \otimes \alpha} := v \otimes \overline{\alpha}$, which corresponds to a base change in the real vector space $\mathfrak{g}_{\mathbb{C}}$.

Claim. $\overline{\mathfrak{a}} \lhd \mathfrak{g}_{\mathbb{C}}$

It is clear that $\overline{\mathfrak{a}}$ is closed under addition, as $\bar{\cdot}$ is additive. If $x \in \mathfrak{a}$ and $\alpha \in \mathbb{C}$, then $\overline{\alpha} x \in \mathfrak{a}$ by assumption, hence $\alpha \overline{x} = \overline{\overline{\alpha} x} \in \overline{\mathfrak{a}}$. The fact that $\overline{\mathfrak{a}}$ is an ideal follows from the calculation of $[x, \overline{y}] = \overline{[\overline{x}, y]}$ for all $x, y \in \mathfrak{g}_{\mathbb{C}}$ together with the assumption that $\mathfrak{a}$ was an ideal. It follows in particular, that $\mathfrak{a} \cap \overline{\mathfrak{a}} \lhd \mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{a} + \overline{\mathfrak{a}} \lhd \mathfrak{g}_{\mathbb{C}}$ with the additional property of being invariant under the conjugation $\bar{\cdot}$.

We introduce a real linear map $\psi : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ by $\psi(x) := x + \overline{x}$ for all $x \in \mathfrak{g}_{\mathbb{C}}$ and we denote by $\imath : \mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}$ the embedding of $\mathfrak{g}$ as a subalgebra via $\imath(v) := v \otimes 1$ for all $v \in \mathfrak{g}$.

Claim. Let $\mathfrak{b} \lhd \mathfrak{g}_{\mathbb{C}}$, then $\psi(\mathfrak{b}) \lhd \imath(\mathfrak{g})$. If $\overline{\mathfrak{b}} \subseteq \mathfrak{b}$, then $\mathfrak{b} = \psi(\mathfrak{b})_{\mathbb{C}}$.

It is clear that $\psi(\mathfrak{b}) \subseteq \imath(\mathfrak{g})$. Note that for all $v \in \mathfrak{g}$ holds $\overline{\imath(v)} = \imath(v)$, thus by the above calculation:

$$[\imath(v), \psi(x)] = [\imath(v), x] + [\imath(v), \overline{x}] = [\imath(v), x] + \overline{[\imath(v), x]} = \psi\big([\imath(v), x]\big) \in \psi(\mathfrak{b})$$

For the second statement, we note that:

$$x = \frac{1}{2}\psi(x) - \frac{\mathbf{i}}{2}\psi(\mathbf{i}x) \quad \forall x \in \mathfrak{g}_{\mathbb{C}}$$

Thus $\mathfrak{b} \subseteq \psi(\mathfrak{b})_{\mathbb{C}}$. The opposite inclusion is clear: $\alpha\psi(x) = \alpha x + \alpha\overline{x} \in \mathfrak{b}$ for all $x \in \mathfrak{b}$.

We apply this to the ideals $\mathfrak{a} \cap \overline{\mathfrak{a}}$ and $\mathfrak{a} + \overline{\mathfrak{a}}$. Note that $\mathfrak{b} = \overline{\mathfrak{b}}$ implies that $\psi(\mathfrak{b}) \subseteq \mathfrak{b}$, hence if $\psi(\mathfrak{a} \cap \overline{\mathfrak{a}}) = \mathfrak{g}$, then $\mathfrak{a} \cap \overline{\mathfrak{a}} = \mathfrak{g}_{\mathbb{C}}$, contradicting our initial assumption on $\mathfrak{a}$ and similarly if $\psi(\mathfrak{a} + \overline{\mathfrak{a}}) = \{0\}$, then $\mathfrak{a} \subseteq \mathfrak{a} + \overline{\mathfrak{a}} = \psi(\mathfrak{a} + \overline{\mathfrak{a}})_{\mathbb{C}} = \{0\}$, yielding another contradiction. It follows that $\mathfrak{a} \cap \overline{\mathfrak{a}} = \{0\}$ and $\mathfrak{a} + \overline{\mathfrak{a}} = \mathfrak{g}_{\mathbb{C}}$ and thus $\mathfrak{g}_{\mathbb{C}} = \mathfrak{a} \oplus \overline{\mathfrak{a}}$ as Lie algebras. This implies by a dimension argument that $\mathfrak{a} \cong \mathfrak{g}$ as $\mathbb{R}$-vector spaces. On the other hand $\mathfrak{a} \cap \overline{\mathfrak{a}} = \{0\}$ yields that ψ is an isomorphism of Lie algebras and thus $\mathfrak{g}$ carries a complex structure compatible with its real structure, making it a complex, simple Lie algebra. It is clear that $\bar{\cdot} : \mathfrak{a} \to \overline{\mathfrak{a}}$ is an isomorphism of real Lie algebras. Note that and it is conjugate linear, hence the two vector spaces are isomorphic as complex vector spaces. □

The above description of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}$ implies that if the original bilinear form σ was $\mathbb{C}$-linear, then we can apply the argument applied to $\mathfrak{g}_{\mathbb{C}}$ before in the case of an absolutely simple Lie algebra to $\mathfrak{g}$ instead and the claim follows. More explicitely: the Lie algebra $\mathfrak{g}$ is a complex, simple Lie algebra and $\sigma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a bilinear form. If σ is degenerate, $\{0\} \neq \mathrm{rad}(\sigma) \lhd \mathfrak{g}$ and thus $\sigma \equiv 0$ by simplicity. If σ is non-degenerate, σ^* and B^* define isomorphisms from $\mathfrak{g}$ to $\mathfrak{g}^*$ and thus $(B^*)^{-1} \circ \sigma^*$ is an automorphism of the vector space $\mathfrak{g}$. As everything is defined over the complex numbers, we can find an eigenvalue λ and an eigenvector x of $(B^*)^{-1} \circ \sigma^*$. This yields $0 \neq x \in \mathrm{rad}(\sigma - \lambda B) \lhd \mathfrak{g}$ and hence $\sigma = \lambda B$. This argument applies in particular to the bilinear form $(X, Y) \mapsto \mathrm{tr}(XY)$. If σ was not $\mathbb{C}$-bilinear, the statement is in general false. We can show this quite generally:

Proposition. *Let $\mathfrak{g}$ be an absolutely simple real Lie algebra, $\mathfrak{h} := \mathfrak{g}_{\mathbb{C}}$ as a real Lie algebra. Let $B_{\mathrm{ad}}(\mathfrak{h})$ be the vector space of bilinear, symmetric forms on $\mathfrak{h}$ for which $\mathfrak{h}$ is skew-symmetric. Then* $\dim B_{\mathrm{ad}}(\mathfrak{h}) \geq 2$.

Proof. Note that as shown above $\mathfrak{h}$ is not absolutely simple and that $\mathfrak{h}_{\mathbb{C}} \cong \mathfrak{a} \oplus \overline{\mathfrak{a}}$, where $\mathfrak{a} \cong \mathfrak{g}_{\mathbb{C}}$ and $\overline{\mathfrak{a}}$ is conjugate isomorphic to $\mathfrak{g}_{\mathbb{C}}$. This we can use to induce two linearly independent bilinear forms on $\mathfrak{h}$ as follows: The above argumentation implies that $\mathfrak{a}$ and $\overline{\mathfrak{a}}$ are conjugate isomorphic, i.e. there exists a bijection $\Phi : \mathfrak{a} \to \overline{\mathfrak{a}}$ such that:

$$\Phi(\alpha v + \beta w) = \overline{\alpha}\Phi(v) + \overline{\beta}\Phi(w)$$

which also respects the Lie bracket. Let $B_{\overline{\mathfrak{a}}}$ denote the Killing form on $\overline{\mathfrak{a}}$. Then B given by:

$$B(v, w) := \overline{B_{\overline{\mathfrak{a}}}\big(\Phi(v), \Phi(w)\big)} \quad \forall v, w \in \mathfrak{a}$$

is a non-degenerate, symmetric, $\mathbb{C}$-bilinear form on $\mathfrak{a}$ so that $\mathfrak{a}$ acts via skew-symmetries. In particular – by simplicity of $\mathfrak{a}$ – B is proportional to the Cartan-Killing form on $\mathfrak{a}$. Being $\mathbb{C}$-bilinear implies $\mathbb{R}$-bilinearity and thus B is a non-degenerate, symmetric $\mathbb{R}$-bilinear form on $\mathfrak{h}$ for which $\mathfrak{h}$ acts skew-symmetrically. Define $\tilde{B}$ by:

$$\tilde{B}(v, w) := B_{\overline{\mathfrak{a}}}\big(\Phi(v), \Phi(w)\big) \quad \forall v, w \in \mathfrak{a}$$

Then $\tilde{B}$ is additive, symmetric, non-degenerate and $\mathfrak{a}$ acts via skew-symmetries. However $\tilde{B}$ is *not* $\mathbb{C}$-bilinear but $\mathbb{C}$-conjugate-bilinear, i.e.:

$$\tilde{B}(\alpha v, \beta w) = \overline{\alpha\beta}\tilde{B}(v, w) \quad \forall v, w \in \mathfrak{a} \forall \alpha, \beta \in \mathbb{C}$$

Non-degeneracy implies that B and $\tilde{B}$ are not proportional. Note however that $\tilde{B}$ is $\mathbb{R}$-bilinear and thus $\tilde{B}$ yields a non-degenerate symmetric, $\mathbb{R}$-bilinear form on $\mathfrak{h}$ for which $\mathfrak{h}$ acts by skew-symmetries. This proves the claim. □

Exercise 4

Recall from class that for $K = \mathrm{SO}(2)$, $A = \{\left(\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right); a > 0\}$ and $N = \{\left(\begin{smallmatrix} 1 & t \\ & 1 \end{smallmatrix}\right); t \in \mathbb{R}\}$ holds $\mathrm{SL}_2(\mathbb{R}) = KAN$ and that the product map $K \times A \times N \to KAN$ is a diffeomorphism. On the other hand:

$$\mathrm{PSL}_2(\mathbb{R}) \cong {}_{\{\pm I\}}\backslash^{KAN} \cong \left({}_{\{\pm I\}}\backslash^{K}\right) AN$$

and it is clear that the product map ${}_{\{\pm I\}}\backslash^{K} \times A \times N \to \mathrm{PSL}_2(\mathbb{R})$ is a local diffeomorphism as $\mathrm{PSL}_2(\mathbb{R})$ is locally diffeomorphic to $\mathrm{SL}_2(\mathbb{R})$, but it is also bijective and hence indeed a diffeomorphism. One easily sees that $K \cong {}_{\{\pm I\}}\backslash^{K}$ (e.g. explicitely by angle doubling) and hence follows that $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{R})$ are diffeomorphic. In order to show that $\mathrm{PSL}_2(\mathbb{R})$ is not isomorphic to $\mathrm{SL}_2(\mathbb{R})$, we show that $\mathrm{PSL}_2(\mathbb{R})$ has no center. Assume otherwise, i.e. assume that $g \in \mathrm{SL}_2(\mathbb{R})$ is such that $gxg^{-1}x^{-1} \in \{\pm I\}$ for all $x \in \mathrm{SL}_2(\mathbb{R})$. If g is in the center of $\mathrm{SL}_2(\mathbb{R})$, then $g \in \{\pm I\}$ and in particular $g\{\pm I\}$ is the identity in $\mathrm{PSL}_2(\mathbb{R})$. So assume otherwise, then the requirement implies that there is some $x \in \mathrm{SL}_2(\mathbb{R})$ satisfying $gxg^{-1} = -x$. But conjugation preserves eigenvalues and hence this is absurd. Thus the center of $\mathrm{PSL}_2(\mathbb{R})$ is trivial and hence $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R})$ are non-isomorphic.

E-mail address: manuel.luethi@math.ethz.ch