ETH Zürich HS 2015
D-MATH
Prof. H. Mete Soner

Coordinator
Matti Kiiski

Mathematical Finance
Exercise Sheet 1

Please hand in until Friday, 02.10.2015, 12:00. The box is in the room HG G 53.

Exercise 1-1:

a) Consider a discrete-time model with finite time horizon T and price process $\bar{S} = (S, 1)$, i.e., there is one stock and a bank account.

   i) Describe in mathematical terms the following self-financing strategy: Start with one unit of the stock. Whenever the price of the stock decreases, double your stock holdings (unless you are holding a non-positive number of shares, in which case you increase your position to $+1$). If the price increases, sell three units. If the price remains constant, do not change the portfolio.

   ii) Describe in words the following strategy: Fix $0 < a < b$ and define

   $$\sigma_1 = \inf\{t = 0, 1, \dots, T : S_t \leq a\} \wedge T, \quad \tau_1 = \inf\{t = \sigma_1, \sigma_1 + 1, \dots, T : S_t \geq b\} \wedge T,$$

   and inductively

   $$\sigma_{n+1} = \inf\{t = \tau_n, \tau_n + 1, \dots, T : S_t \leq a\} \wedge T,$$
   $$\tau_{n+1} = \inf\{t = \sigma_{n+1}, \sigma_{n+1} + 1, \dots, T : S_t \geq b\} \wedge T.$$

   Then set

   $$\vartheta_t = \sum_k \mathbf{1}_{\sigma_k < t \leq \tau_k}.$$

b) Now consider a continuous-time model with horizon T and price process $\bar{S} = (S, 1)$. Under suitable assumptions on S, describe in mathematical terms the following strategies:

   i) Invest 20% of the current wealth in the riskless asset and 80% in the stock.

   ii) Fix $K \in \mathbb{R}$. We start with one (zero) share if $S_0 > K$ ($\leq K$). Whenever the stock price falls below K or equals K, the share is sold, and whenever the price returns to a level strictly above K, one share is bought again. The amount held in the riskless asset is given by $\eta_t = -K 1_{\{S_t > K\}}$.

   iii) Which of these strategies is self-financing?

c) Assume that $\bar{S} = (S^1, \dots, S^d, 1)$ is positive and continuous. There are three ways to describe portfolio holdings in the risky assets: the number of shares of each asset held in the portfolio, or the monetary amount invested, or the fraction of wealth invested. How are these quantities related? Are these notions equivalent?

Exercise 1-2:

a) Consider a probability space with discrete-time filtration $(\mathcal{F}_t)_{t=0,1,\ldots}$ and an $\mathbb{R}^d$-valued local martingale $X = (X_t)_{t=0,1,\ldots}$ null at $t = 0$. Prove the following:

   i) If ϑ is any $\mathbb{R}^d$-valued predictable process, the real-valued discrete-time integral $Y = \int \vartheta\, dX$, defined by $Y_t = \sum_{k=1}^t \vartheta_k (X_k - X_{k-1})$, is again a local martingale.

   ii) If all the X_t are integrable, X is a true martingale.

b) If $L : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is a local martingale such that $L \geq -1$, then L is a super-martingale.

Note: Assertions in a) are false in continuous time.

Exercise 1-3:

Consider for the (discounted) stock price process S the continuous-time model under P with time horizon $T = 1$ given by

$$dS_t = S_t \left(\frac{1}{X_t} dt + dW_t \right), \quad S_0 > 0$$

where $W = \{W_t\}_{0 \leq t \leq 1}$ is a standard P-Brownian motion and $X = \{X_t\}_{0 \leq t \leq 1}$ is given by

$$dX_t = \left(\frac{1}{X_t} - 2 \right) dt + dW_t, \quad X_0 = 1.$$

For the bank account, assume that $B_t = 1$ for all $t \in [0, 1]$.

a) Prove that $P(X_t > 0, 0 \leq t \leq 1) = 1$.
Hint: By Girsanov's theorem, there exists a probability measure $\tilde{P}$ equivalent to P such that $\tilde{W}_t := W_t - 2t$ is a $\tilde{P}$-Brownian motion. Using this show that the norm of a 3-dimensional $\tilde{P}$-Brownian motion B with $B_0 = (1, 0, 0)$ coincides in law with

$$X_t = 1 + \int_0^t \frac{1}{X_s} ds + \tilde{W}_t, \quad X_0 = 1.$$

You can further use that the solution of this SDE is unique in law and a 3-dimensional Brownian motion never attains origin.

b) Show that the cumulative gains process $G_t(\varphi) = \int_0^t \theta_u dS_u$ for $\theta_t := \frac{1}{S_t}$ is a.s. bounded from below by -1.

c) Prove that $\theta_t = \frac{1}{S_t}$ is an arbitrage opportunity, that is, a self-financing strategy $\varphi = \left(\eta, \frac{1}{S}\right)$ corresponding to $(V_0, \theta) = \left(0, \frac{1}{S}\right)$ with $V_0 = 0, V_1(\varphi) \geq 0$ P-a.s. and $P[V_1(\varphi) > 0] > 0$.

Exercise 1-4:

First consider a discrete-time financial market with finite time horizon $T < \infty$ consisting of one risky asset and one money market account.
Let $\Omega = \{-1, 1\}^T$. Define the i.i.d. random variables $Z = \{Z_k,\}_{k=0,1,\ldots,T}$ that takes the values ± 1 with probability $P(Z_k = +1) = P(Z_k = -1) = \frac{1}{2}$. Take the filtration $\mathbb{F}$ generated by the random variables $\{Z_k\}_{k=0,1,\ldots,T}$, i.e. for all $k \geq 0$, $\mathcal{F}_k = \sigma(Z_i; 0 \leq i \leq k)$.

We take the money market account with price process $B_k = 1$ for $k = 0, 1, \dots, T$. The discounted price process of the risky asset $\{S_k\}_{k=0,1,\dots,T}$ is a simple random walk, i.e. $S_0 = 0$ and $S_k = \sum_{i=0}^{k} Z_i$.

Consider the following strategy: Start with zero initial wealth and bet on $+1$ and keep doubling your bets until the first time $\{Z_k\}_{k=0,1,\dots,T}$ takes the value $+1$.

a) Find the self-financing strategy $\varphi = (\eta, \theta)$ and the associated wealth process $V = \{V_k(\varphi)\}_{k=0,1,\dots,T}$ with zero initial wealth for this strategy.

b) Show that V is a martingale and calculate $P(\tau \le t)$ for $\tau = \inf\{k \in \mathbb{N} : S_k - S_{k-1} = 1\}$.

c) Now consider the same financial market with infinite time horizon. Show that with this strategy, $V_\infty(\varphi) := \lim_{t\to\infty} V_t(\varphi) = 1$ a.s.. Explain why one has to borrow huge amounts until one wins.

Exercise 1-5:

Let $V(\varphi)$ be as in part c) of previous exercise 1-4. Put

$$X_t = \begin{cases} V_k(\varphi), & \text{if } 1 - \frac{1}{k+1} \le t < 1 - \frac{1}{k+2}, \\ 1, & \text{if } t \ge 1. \end{cases}$$

Let B be a standard Brownian motion and $\tau = \inf\{t \ge 0 : B_t = 1\}$. Put

$$Y_t = \begin{cases} B^\tau_{t/(1-t)}, & \text{if } t < 1, \\ 1, & \text{if } t \ge 1. \end{cases}$$

Assume the natural filtration and answer the following questions:

a) Is $X = (X_t)_{t\ge0}$ a martingale? Is it a local martingale?

b) Is $Y = (Y_t)_{t\ge0}$ a martingale? Is it a local martingale?

Justify your answers.
