ETH Zürich HS 2015
D-MATH
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Mathematical Finance
Exercise Sheet 4

Please hand in until Friday, 6.11.2015, 12:00.

Exercise 4-1

Consider a financial market with a constant interest rate r and a risky asset whose discounted price process follows a Merton-type jump diffusion, that is,

$$\frac{dS_t}{S_{t-}} = (\mu - r)dt + \sigma dW_t + dJ_t, \quad S_0 > 0,$$

where J is a compensated compound Poisson process with jumps $\Delta J \geq 0$ and independent of W, and ν, σ are constant. More precisely,

$$J_t = \sum_{k=1}^{N_t}(Y_k - 1),$$

where N is a Poisson process with intensity λ and $Y_k \geq 1$ are integrable i.i.d. random variables with $m := E[Y_1]$. The filtration is $\mathbb{F} = \mathbb{F}^{W,J}$. Show that the process S can be expressed as

$$S_t = S_0 \exp(\tilde{X}_t),$$

where $\tilde{X}$ is a Lévy process.

Exercise 4-2

Consider a stock price model based on a 2-dimensional Brownian motion (W, W') in its own filtration, where 1-dimensional stock price follows

$$dS_t = \mu_t dt + \sigma_t dW_t$$

and $\mu_t = \mu(t, S_t, Y_t)$ and $\sigma_t = \sigma(t, S_t, Y_t)$ are determined by continuous functions depending on the diffusion Y which follows

$$\begin{aligned} dY_t &= b(t, Y_t)dt + a(t, Y_t)dB_t, \\ Y_0 &= y_0. \end{aligned}$$

Here B is a correlated Brownian motion: $B_t = \rho W_t + \sqrt{1-\rho^2}W_t'$ for some constant $\rho \in (0,1)$. We assume that the function $\frac{\mu}{\sigma}$ is uniformly bounded. The process Y is often called stochastic factor in this context.

a) Show that $d\langle B, W\rangle_t = \rho dt$ and $d\langle S, Y\rangle_t = a(t, Y_t)\sigma(t, S_t, Y_t)\rho dt$.

b) What is the general form of the density process of an ELMM Q for S?

c) Find the dynamics of S and Y under such a measure; that is give stochastic differential equations for these processes involving only Brownian motions under Q, not under P.

Exercise 4-3

Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, P)$ be a filtered probability space with $\mathcal{F}_0$ P-trivial and $(\mathcal{F})_{t\in[0,T]}$ satisfying the usual conditions, and let $S = (S_t^1, \dots, S_t^d)_{t\in[0,T]}$ be an $\mathbb{R}^d$-valued semimartingale satisfying NFLVR. Denote by $L_+^0(\Omega, \mathcal{F}, P)$ the space of all random variables taking values in $[0, \infty]$, endowed with the topology of convergence in probability. For $x > 0$ and $y > 0$, define the sets

$$\mathcal{C}(x) := \{X_T \in L_+^0(\Omega, \mathcal{F}, P) : X_T \leq x + (H \cdot S)_T \text{ for admissible } H\}$$
$$\mathcal{D}(y) := y\mathcal{D}(1) := y\mathcal{D} := y\{Y_T \in L_+^0(\Omega, \mathcal{F}, P) : E[X_T Y_T] \leq 1 \ \forall X_T \in \mathcal{C}(1)\}.$$

Moreover, let $U :]0, \infty[\to \mathbb{R}$ be an increasing and concave (utility) function, and suppose that $u(x) := \sup_{X_T\in\mathcal{C}(x)} E[U(X_T)] < \infty$ for some (and hence all) $x > 0$. Set $U^+(\mathcal{C}(x)) := \{U^+(X_T) : X_T \in \mathcal{C}(x)\}$.

a) Fix $x > 0$. Show that $\mathcal{C}(x)$ is convex and closed in $L_+^0(\Omega, \mathcal{F}, P)$.

   Hint: Use the fact that $X_T \in L_+^0(\Omega, \mathcal{F}, P)$ is in $\mathcal{C}(x)$ if and only if $\sup_{Y_T\in\mathcal{D}} E[X_T Y_T] \leq x$.

b) Fix $x > 0$. Suppose that that $U^+(\mathcal{C}(x))$ is uniformly integrable. Show that there exists $\widehat{X} \in \mathcal{C}(x)$ such that $E[U(\widehat{X})] = u(x)$.

   Hint: Part (a) and Komlos' Lemma.

c) Suppose now that there exist $a > 0$ and $0 < b < 1$ such that $U^+(x) \leq a(1 + x^b)$ for all $x > 0$ and an equivalent σ-martingale measure $Q \sim P$ on $\mathcal{F}$ for S such that $\left(\frac{dQ}{dP}\right)^{-1}$ has moments of all orders. Fix $x > 0$, and show that $U^+(\mathcal{C}(x))$ is uniformly integrable.

   Hint: By using the growth assumption on U^+, reduce the problem to showing that $\mathcal{C}(x)$ is bounded in L^p with $p > 1$ small enough. Then switch from P to Q.

Exercise 4-4

Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, P)$ be a filtered probability space with $\mathcal{F}_0$ P-trivial and $(\mathcal{F})_{t\in[0,T]}$ satisfying the usual conditions and let $S = (S_t^1, \dots, S_t^d)_{t\in[0,T]}$ be an $\mathbb{R}^d$-valued semimartingale satisfying NFLVR. Assume that there exists a unique equivalent σ-martingale measure $Q \sim P$ on $\mathcal{F}_T$, i.e., the market $(1, S)$ is complete. Let $U :]0,\infty[\to \mathbb{R}$ be an increasing differentiable strictly concave utility function such that

$$U'(0) := \lim_{x\downarrow 0} U'(x) = \infty.$$

Define the functions u, V, v and I on $]0,\infty[$ as in the lecture and the sets $\mathcal{C}(x)$ and $\mathcal{D}(y)$, $x, y > 0$, as in the previous exercise. We assume that $u(x) < \infty$ for some (and hence all) $x \in]0,\infty[$.

a) Fix $y > 0$. Show that

$$z \le y\frac{dQ}{dP} \quad P\text{-a.s. for all } z \in \mathcal{D}(y),$$

where $\frac{dQ}{dP}$ denotes the density of Q with respect to P on $\mathcal{F}$. Deduce that

$$v(y) = \inf_{z\in\mathcal{D}(y)} E[V(z)] = E\left[V\left(y\frac{dQ}{dP}\right)\right],$$

where $E[V(z)] := \infty$ if $V^+(z) \notin L^1(P)$.

Hint: Suppose that there is $z \in \mathcal{D}(y)$ such that $A := \{z > y\frac{dQ}{dP}\}$ has $P[A] > 0$. Set $a := Q[A]$, and use that completeness of S is equivalent to the *predictable representation property* of S under Q to deduce that there exists an admissible H such that $1_A = a + (H \cdot S)_T$.

b) Let $y_0 := \inf\{y > 0 : v(y) < \infty\}$. Show that the function v is in $C^1(]y_0,\infty[)$ and satisfies

$$v'(y) = E\left[\frac{dQ}{dP}V'\left(y\frac{dQ}{dP}\right)\right], \quad y \in]y_0,\infty[.$$

Hint: Apply the fundamental theorem of calculus to the function $y \mapsto V\left(y\frac{dQ}{dP}\right)$ and take expectations.

c) Set $x_0 := \lim_{y\downarrow\downarrow y_0} -v'(y)$. Fix $x \in]0, x_0[$. Let $y_x \in]y_0,\infty[$ be the unique number such that $-v'(y_x) = x$. Show that $\widehat{X} := I\left(y_x\frac{dQ}{dP}\right)$ is the unique solution to the primal problem

$$u(x) = \sup_{X\in\mathcal{C}(x)} E[U(X)].$$

Hint: Show that $\widehat{X} \in \mathcal{C}(x)$ using part (a). Then use a Taylor expansion and the strict concavity of U in $]0,\infty[$ to argue that we have $E[U(X) - U(\widehat{X})] \le 0$ for all $X \in \mathcal{C}(x)$ and that the inequality is an equality if and only if $X = \widehat{X}$ P-a.s..

Exercise 4-5

Let $(\Omega, \mathcal{F}, P)$ be a probability space supporting a Brownian motion $W = (W_t)_{t\in[0,T]}$. Denote by $(\mathcal{F}_t^W)_{t\in[0,T]}$ the natural (completed) filtration of W. Let $\sigma > 0$ and $\mu, r \in \mathbb{R}$. Consider the *undiscounted* Black-Scholes market $(\widetilde{S}^0, \widetilde{S}^1) = (\widetilde{S}_t^0, \widetilde{S}_t^1)_{t\in[0,T]}$ given by the SDEs

$$d\widetilde{S}_t^0 = r\widetilde{S}_t^0 dt, \quad \widetilde{S}_0^0 = 1, \quad \text{and} \quad d\widetilde{S}_t^1 = \widetilde{S}_t^1(\mu dt + \sigma dW_t), \quad \widetilde{S}_0^1 = s > 0.$$

Denote by $S^1 := \frac{\widetilde{S}^1}{\widetilde{S}^0}$ the discounted stock price. Let $U :]0,\infty[\to \mathbb{R}$ be defined by $U(x) = \frac{1}{\gamma}x^\gamma$, where $\gamma \in]-\infty, 1[\setminus\{0\}$.

a) Using part (a) of previous exercise show that

$$v(y) = \frac{1-\gamma}{\gamma} y^{-\frac{\gamma}{1-\gamma}} \exp\left(\frac{1}{2}\frac{\gamma}{(1-\gamma)^2}\frac{(\mu-r)^2}{\sigma^2}T\right), \quad y \in]0,\infty[.$$

b) Using part (c) of previous exercise show that $\widehat{X} := x\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu-r}{\sigma}R\right)_T$, where $R = (R_t)_{t\in[0,T]}$ is given by $R_t = W_t + \frac{\mu-r}{\sigma}t$, is the unique solution to the primal problem

$$u(x) = \sup_{X\in\mathcal{C}(x)} E[U(X)], \quad x \in]0,\infty[.$$

c) Deduce that $\widehat{X}_x$ is the terminal wealth gained from the initial capital x trading according to θ^x, where $\theta^x = (\theta_t^x)_{t\in[0,T]}$ is given by

$$\theta_t^x = \frac{x}{S_t}\frac{1}{1-\gamma}\frac{\mu-r}{\sigma^2}\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu-r}{\sigma}R\right)_t, \quad x \in]0,\infty[,$$

and show that

$$u(x) = \frac{x^\gamma}{\gamma}\exp\left(\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}T\right), \quad x \in]0,\infty[.$$