ETH Zürich HS 2015
D-MATH
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Mathematical Finance
Exercise Sheet 5

Please hand in until Friday, 20.11.2015, 12:00.

Consider a finite horizon model $T < \infty$ with one risky and one risk-free asset. The price of the risky asset S is governed by geometric Brownian motion with volatility σ and drift μ. The bond price B has a constant interest rate $\mu > r$. An agent invests at any time t a proportion π_t of his wealth in the stock and $1 - \pi_t$ in the bond. The wealth process X evolves according to

$$\frac{dX_s}{X_s} = r ds + \pi_s \left((\mu - r) ds + \sigma dW_s \right), \quad X_0 = x. \tag{1}$$

Let $\mathcal{A}$ be the set of all progressively measurable processes π taking values in a closed convex set $A \subseteq \mathbb{R}$ such that the SDE (1) has a unique strong solution that satisfies

$$E\left[\sup_{t \le s \le T} |X_s^{t,x}|^2\right] < \infty \text{ for all } 0 \le t \le T.$$

The agent wants to maximize the expected utility from terminal wealth at horizon T. Take a nondecreasing and concave utility function U on $\mathbf{R}_+$. The value function of the utility maximization problem is then defined by

$$u(t,x) = \sup_{\pi \in \mathcal{A}} E\left[U(X_T^{t,x})\right], \quad (t,x) \in [0,T] \times \mathbf{R}_+.$$

Exercise 5-1

a) Show that for all $t \in [0,T]$, $u(t,\cdot)$ is also increasing and concave in x.

b) Write down the dynamic programming principle and the Hamilton-Jacobi-Bellman equation for this stochastic control problem and specify the terminal condition.

Exercise 5-2

Now assume that the utility function in the previous exercise 5-1 b) is of the form

$$U(x) = \frac{x^\gamma}{\gamma} \quad x \ge 0, \ 0 \ne \gamma < 1.$$

Look for a candidate solution of the form $w(t,x) = \phi(t)U(x)$ and also specify a candidate for the optimal portfolio process.

Exercise 5-3

Consider the same setup in the previous question with $U(x) = \frac{x^\gamma}{\gamma}$ for $x \geq 0$ with $0 \neq \gamma < 1$. Show by a verification argument that the value function to the utility maximization problem is equal to the candidate solution in the previous exercise 5-2 and the optimal control agrees with the candidate optimal portfolio process.