ETH Zürich HS 2015
D-MATH
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Mathematical Finance
Exercise Sheet 7

Please hand in until Friday, 18.12.2015, 12:00.

Consider a finite horizon model $T < \infty$ with one risky and one risk-free asset. The price of the risky asset S is governed by a geometric Brownian motion with volatility σ and drift μ. The bond price B is supposed to be normalized: $B \equiv 1$. Assume that selling a risky asset is subject to a transaction cost $\lambda \in]0, 1[$. Take a strictly increasing and strictly concave utility function U on $\mathbb{R}$. Given an initial endowments $x, y \in \mathbb{R}$, the utility maximization problem is

$$\sup_{\varphi \in \mathcal{A}} E\left[U(X_T^\varphi)\right], \tag{1}$$

where X_T^φ is the liquidation value of the portfolio (φ^0, φ) at time T:

$$X_T^\varphi = \varphi_T^0 + \varphi_T^+(1-\lambda)S_T - \varphi_T^- S_T.$$

Each $\varphi \in \mathcal{A}$ is assumed to be of finite variation, bounded, and self-financing, i.e.,

$$d\varphi_t^0 = -S_t d\varphi_t^b + (1-\lambda)S_t d\varphi_t^s,\ t \in [0,T],\ (\varphi_{0-}^0, \varphi_{0-}) = (x, y),$$

where $\varphi_t = \varphi_t^b - \varphi_t^s$ is the difference of cumulative number of shares bought (φ_t^b) and sold (φ_t^s) at time t.

Exercise 7-1

a) Write the dynamics for the amount of money invested in the safe asset at time t, η_t^0, and in the risky asset, $\eta_t = \varphi_t S_t$.

b) Derive the HJB equation for (1) using Ito's formula and the martingale optimality principle.

Exercise 7-2

A strategy (φ^0, φ) is called long-term optimal if it maximizes the equivalent annuity

$$\liminf_{T\to\infty} \frac{1}{T} U^{-1}(E[U(X_T^\varphi)]).$$

Let $U(x) = -e^{-\alpha x},\ x \in \mathbb{R},\ \alpha > 0$. Make an educated guess that the value function is of the form

$$u(t, \eta_t^0, \eta) = -e^{-\alpha\eta_t^0} e^{-\alpha\beta\eta_t^0} \phi(\eta_t),$$

for some $\beta \in \mathbb{R}$ and $\phi : [0,T] \to \mathbb{R}$. Show that the no-trade region for the long-term optimal solution is $]\frac{\mu}{\alpha\sigma} - c, \frac{\mu}{\alpha\sigma} + c[$, where $c = \frac{1}{\alpha}\sqrt{\mu^2/\sigma^4 - 2\beta/(\alpha\sigma^2)}$.

Exercise 7-3

It can be further argued that ϕ in the previous exercise is of the form

$$\phi(\eta_t) = e^{-\int_0^{\log(\eta_t)/\eta_{\alpha-}} w(y)dy},$$

where the function $w : [0, \log(\frac{1}{1-\lambda}\frac{\eta_{\alpha+}}{\eta_{\alpha-}})] \to [\frac{\mu}{\sigma} - \alpha c, \frac{\mu}{\sigma} + \alpha c]$ is increasing and surjective, $w = w'$ on the boundaries, and satisfies the ODE

$$w' + w^2 + (2\frac{\mu}{\sigma^2} - 1)w + \frac{\alpha\beta}{\sigma^2} = 0.$$

Let

$$\widetilde{S}_t := \frac{\partial_{\varphi_t} u}{\partial_{\varphi_t^0} u}.$$

Show that $\widetilde{S}$ is an Ito process with the dynamics

$$\frac{d\widetilde{S}_t}{\widetilde{S}} = \sigma^2 w'(\log(\eta_t/\eta_{\alpha-}))dt + \sigma \frac{w'(\eta_t/\eta_{\alpha-})}{w(\eta_t/\eta_{\alpha-})} dW_t$$

taking values on the bid-ask spread $[(1-\lambda)S, S]$.

Exercise 7-4

Show that the strategy $\widehat{\eta}_t := \frac{1}{\alpha} w(\log(\eta_t/\eta_{\alpha-}))$ is long-term optimal with the equivalent annuity β in the *frictionless* market with the price process $\widetilde{S}$.

Exercise 7-5

Show that the strategy $\widehat{\eta}$ is long-term optimal with the equivalent annuity β in the original market with a transaction cost.