ETH Zürich HS 2015
D-MATH
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Mathematical Finance
Solutions Sheet 1

Solution 1-1

a) i) $\vartheta_0 = 1$
$\vartheta_{t+1} = 1_{\{S_t<S_{t-1}\}}(2\vartheta_t 1_{\{\vartheta_t>0\}} + (\vartheta_t+1)1_{\{\vartheta_t\le 0\}}) + 1_{\{S_t>S_{t-1}\}}(\vartheta_t - 3) + 1_{\{S_t=S_{t-1}\}}\vartheta_t$

ii) Buy one share when the price falls below a and sell it if the price reaches b.

b) We assume that S is a positive continuous semimartingale.

i) If V is the corresponding wealth process, we want to set $\eta_t = 0.2V_t$ and $\vartheta_t = 0.8V_t/S_t$. To define V, we use the stochastic exponential. Recall that the unique solution to the equation

$$Z_t = 1 + \int_0^t Z_u \, dM_u$$

is $Z = \mathcal{E}(M)$. So,

$$V_t = V_0 + \int_0^t 0.8 V_u S_u^{-1} \, dS_u$$

for $V_t := V_0 \cdot \mathcal{E}(\int 0.8 S_u^{-1} \, dS_u)_t$.

ii) The holdings in the risky asset are given by $\vartheta_t = 1_{\{S_t>K\}}$.

iii) In the first case, the self-financing condition holds by definition: $\varphi_t \cdot \bar{S}_t = 0.2V_t + 0.8V_t S_t^{-1} S_t = V_t$. The second case is tricky. We first note that $\varphi_t \cdot \bar{S}_t = (S_t - K)^+$ and the initial cost is $(S_0-K)^+$. So the self-financing condition is

$$(S_t - K)^+ = (S_0 - K)^+ + \int_0^t 1_{\{S_u>K\}} \, dS_u.$$

By the fundamental theorem of calculus, this is true if S is absolutely continuous, but the identity is not true for general S. The so-called Tanaka-formula shows that

$$(S_t - K)^+ = (S_0 - K)^+ + \int_0^t 1_{\{S_u>K\}} \, dS_u + \Lambda_t(K),$$

where $\Lambda \geq 0$ is a so-called local time, and Λ vanishes only if S has vanishing quadratic variation. Hence, the strategy is not self-financing whenever there is a nontrivial martingale part in S. See Carr & Jarrow 1990, Rev. Fin. Studies, for more on this so-called *stop-loss start-gain* paradox.

c) If ϑ^i is the number of shares, $\vartheta^i S^i$ is the monetary amount and $\vartheta^i S^i/V$ is the fraction of wealth. There is an obvious problem the pass between the notions when the price can become zero. Moreover, the last notion is less general because in this case we can only allow positive wealth V. (On the other hand, this terminology allows easy access to the riskless holding, which can be important e.g. to formulate "no shorting" constraints for ϑ.)

Solution 1-2

a) The following claims hold only in discrete time.

i) There is a localizing sequence σ^n such that X^{σ^n} is a martingale for all n. For every $n \in \mathbb{N}$ we define the stopping time $\tau^n = \inf\{k \geq 0 : |\vartheta_{k+1}| \geq n\}$ and then

$$E[\vartheta_{t+1}^{\tau^n}(X_{t+1}^{\sigma^n \wedge \tau^n} - X_t^{\sigma^n \wedge \tau^n})|\mathcal{F}_t] = \vartheta_{t+1}^{\tau^n} E[(X_{t+1}^{\sigma^n \wedge \tau^n} - X_t^{\sigma^n \wedge \tau^n})|\mathcal{F}_t] = 0,$$

so $(\vartheta \cdot X^{\sigma^n})^{\tau^n}$ is a martingale, and $(\vartheta \cdot X^{\sigma^n})$ is a local martingale.

ii) Let (τ^m) be a localizing sequence. Note that $\sup_m |X_t^{\tau^m}| \leq \sum_{k=0}^t |X_k| \in L^1$, hence dominated convergence allows passage from $E[X_t^{\tau^m}|\mathcal{F}_{t-1}] = X_{t-1}^{\tau^m}$ to $E[X_t|\mathcal{F}_{t-1}] = X_{t-1}$.

b) Let $T_n \uparrow \infty$ be a localizing sequence for L and let $U \leq V$ be finite stopping times. For each n the process L^{T_n} is a uniformly integrable martingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$. For each $A \in \mathcal{F}_U$ and each $n \leq m$ we therefore have $\int_{A \cap \{U \leq T_n\}} L_{U \wedge T_n} = \int_{A \cap \{U \leq T_n\}} L_{V \wedge T_m}$. Hence $\int_{A \cap \{U \leq T_n\}} L_U = \int_{A \cap \{U \leq T_n\}} L_{V \wedge T_m}$. If we let $m \to \infty$, observe that $L_{V \wedge T_m} \geq -1$ and use Fatou's lemma to obtain that $\int_{A \cap \{U \leq T_n\}} L_U \geq \int_{A \cap \{U \leq T_n\}} L_V = \int_{A \cap \{U \leq T_n\}} E[L_V|\mathcal{F}_U]$. Hence on $U \leq T_n$ we have $L_U \geq E[L_V|\mathcal{F}_U]$. We now let n tend to ∞ to conclude.

Solution 1-3

a) We can define a probability measure $\tilde{P}$ equivalent to P by

$$\left.\frac{d\tilde{P}}{dP}\right|_{\mathcal{F}_t} = \mathcal{E}(2W)_t = \exp(2W_t - 2t),$$

where $\mathcal{F}_t$ denotes the natural filtration of W_t, satisfying the usual conditions. According to Girsanov's theorem $\tilde{W}_t = W_t - 2t$ is then a $\tilde{P}$-Brownian motion. Define now a process $X_t = \|B_t\|$ with $B_0 = (1, 0, 0)$. Since B_t never hits the origin a.s., we can apply Ito's formula to obtain

$$dX_t = \sum_{i=1}^{3} \frac{B_t^i}{\|B_t\|} dB_t^i + \frac{1}{X_t} dt.$$

By Levy's characterization theorem

$$\int_0^t \sum_{i=1}^{3} \frac{B_s^i}{\|B_s\|} dB_s^i$$

coincides in law with the one-dimensional $\tilde{P}$-Brownian motion $\tilde{W}_t$ such that we can write

$$dX_t = \frac{1}{X_t} dt + d\tilde{W}_t.$$

As B_t never hits the origin a.s., it follows that $\tilde{P}[X_t > 0, \forall 0 \leq t \leq 1] = 1$. Under P, we then have the dynamics for X_t

$$dX_t = \left(\frac{1}{X_t} - 2\right) dt + dW_t.$$

As P and $\tilde{P}$ are equivalent, we can deduce that $P[X_t > 0, \forall 0 \leq t \leq 1] = 1$.

b) The cumulative gains process $G_t(\varphi) = \int_0^t \theta_t dS_t$ for $\theta_t = \frac{1}{S_t}$ satisfies

$$dG_t = \frac{1}{X_t}dt + dW_t, \quad G_0 = 0.$$

Thus, $G_t = X_t + 2t - 1$ which implies by (a), $P[G_t \geq -1, \forall 0 \leq t \leq 1] = 1$.

c) We can choose η so that $\varphi = (\eta, \theta)$ is self-financing. Then we have $V_t = 0 + \int_0^t \theta_u dS_u = G_t$. Thus $V_1 = G_1 = X_1 + 1 \geq 1$ a.s.

Solution 1-4

a) Define the stopping time with respect to the filtration $\mathcal{F}$

$$\tau(\omega) = \inf\{k \in \mathbf{N} : S_k - S_{k-1} = 1\}.$$

The trading strategy θ is given by

$$\theta_1 = 1, \quad \theta_k = 2^{k-1}\mathbf{1}_{\{\tau > k-1\}}.$$

Since τ is a stopping time, θ is predictable. Consider the self-financing strategy $\varphi = (\eta, \theta)$ associated to $(V_0, \theta) = (0, \theta)$ and

$$\begin{aligned}
V_t(\varphi) &= \sum_{k=1}^{t} 2^{k-1}\mathbf{1}_{\{\tau > k-1\}}(S_k - S_{k-1}) \\
&= \mathbf{1}_{\{\tau > t\}} \sum_{k=1}^{t} 2^{k-1}(-1) + \mathbf{1}_{\{\tau \leq t\}} \sum_{k=1}^{\tau} 2^{k-1}\mathbf{1}_{\{\tau > k-1\}}(S_k - S_{k-1}) \\
&= \mathbf{1}_{\{\tau > t\}}(1 - 2^t) + \mathbf{1}_{\{\tau \leq t\}} \left(\sum_{k=1}^{\tau-1} 2^{k-1}(-1) + 2^{\tau-1}(+1) \right) \\
&= \mathbf{1}_{\{\tau > t\}}(1 - 2^t) + \mathbf{1}_{\{\tau \leq t\}}1.
\end{aligned}$$

b) Since θ is bounded for finite time horizon $T < \infty$, V_t is integrable for all $t \in \{0, \dots, T\}$. Since θ is predictable, it follows by the martingale property that

$$\begin{aligned}
E[V_t(\varphi)|\mathcal{F}_{t-1}] &= E\left[\sum_{k=1}^{t} \theta_k (S_k - S_{k-1}) \middle| \mathcal{F}_{t-1} \right] \\
&= V_{t-1} + \theta_t E[S_t - S_{t-1} \mid \mathcal{F}_{t-1}] = V_{t-1}(\varphi).
\end{aligned}$$

Since $V_t(\varphi)$ is a martingale and $V_0 = 0$, we obtain

$$0 = E[V_t(\varphi)] = (1 - 2^t)P(\tau > t) + P(\tau \leq t).$$

So solving for $P(\tau \leq t) = \frac{2^t - 1}{2^t}$. Letting $t \to \infty$ yields that $P(\tau < \infty) = 1$.

c) Since $P(\tau < \infty) = 1$, we obtain that

$$V_\infty(\theta) = \lim_{t \to \infty} V_t(\theta) = 1 \quad P - a.s.$$

Because $V_t(\varphi) = \mathbf{1}_{\{\tau > t\}}(1 - 2^t) + \mathbf{1}_{\{\tau \leq t\}}1$, if one wins up to time t, one gets $+1$, otherwise one has lost $2^t - 1$. Therefore, until one eventually wins, one may need to borrow huge amounts of money.

Solution 1-5

a) Since $EX_1 = 1 \neq EX_0 = 0$, X is not a martingale. However, it is a local martingale. Choose $\tau_n = \inf\{t \geq 0 : |X_t| \geq n\} \wedge n$, $n \geq 1$. Let $s < t < 1$ and $A \in \mathcal{F}_s$. We have

$$E[1_A(X_t^{\tau_n} - X_s^{\tau_n})] = E[1_A(X_{(\tau^n \wedge t) \vee s} - X_s)].$$

By the optional stopping theorem, $(X_t^{\tau_n})_{t \geq 0}$ is a uniformly bounded martingale on $t < 1$. Moreover, X^{τ_n} is continuous at $t = 1$, and constant on $t \geq 1$. Therefore, $(X_t^{\tau_n})_{t \geq 0}$ is martingale, for every $n \geq 1$. This shows that X is a local martingale.

b) By the same arguments, Y is not a martingale, but it is a local martingale.