ETH Zürich HS 2015
D-MATH
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Mathematical Finance
Solutions Sheet 2

Solution 2-1

Suppose that θ is a Type 2 arbitrage and that $V_t(\theta)$ is **not** non-negative a.s. for at least one $t \in \{0, 1, ..., T-1\}$. Then $\exists\ t < T$ and $A \in \mathcal{F}_t$ with $P(A) > 0$ s.t.

$$(\theta \cdot S)_t(w) < 0 \text{ for } \omega \in A,$$
$$(\theta \cdot S)_u \geq 0 \text{ a.s. for } u > t.$$

We amend θ to a new strategy ϕ by setting $\phi_u(\omega) = 0$ for all $u \in \mathbf{T}$ and $\omega \in \Omega \setminus A$, while on A we set $\phi_u(\omega) = 0$ if $u \leq t$, and for $u > t$ we define

$$\phi_u^0(\omega) = \theta_u^0(\omega) - \frac{\theta_t \cdot S_t}{S_t^0(\omega)}$$
$$\phi_u^i(\omega) = \theta_u^i(\omega) \quad \text{for} \qquad i = 1, 2, ...d.$$

This strategy is obviously predictable. It is also self-financing: On $\Omega \setminus A$ we have $V_u(\phi) \equiv 0$ for all $u \in \mathbf{T}$, while on A we need only check that $\Delta\phi_{t+1} \cdot S_t = 0$, since $\Delta\theta_u$ and $\Delta\phi_u$ differ only for $u = t+1$. We observe that $\phi_t^i = 0$ on $\Omega \setminus A$ for $i > 0$ and that, on A

$$\Delta\phi_{t+1}^0 = \phi_{t+1}^0 = \theta_{t+1}^0 - \frac{\theta_t \cdot S_t}{S_t^0},$$
$$\Delta\phi_{t+1}^i = \theta_{t+1}^i \qquad \text{for} \qquad i = 1, 2, ..., d.$$

Since θ is self-financing

$$(\Delta\phi_{t+1}) \cdot S_t = 1_A(\theta_{t+1} \cdot S_t - \theta_t \cdot S_t) = 1_A(\theta_t \cdot S_t - \theta_t \cdot S_t) = 0.$$

Now we show that $V_u(\phi) \geq 0$ for all $u \in \mathbf{T}$ and $P(V_T(\phi) > 0) > 0$. First note that $V_u(\phi) = 0$ on $\Omega \setminus A$ for all $u \in \mathbf{T}$. On A we also have $V_u(\phi) = 0$ when $u \leq t$, but for $u > t$ we obtain

$$V_u(\phi) = \phi_u \cdot S_u = \big(\theta_u^0 - \frac{\theta_t \cdot S_t}{S_t^0}\big)S_u^0 + \sum_{i=1}^{d} \theta_u^i \cdot S_u^i = \theta_u \cdot S_u - \frac{\theta_t \cdot S_t}{S_t^0} S_u^0$$

We have $(\theta \cdot S)_u \geq 0$ for $u > t$, and $(\theta \cdot S)_t < 0$ while $S_t^0 > 0$, it follows that $V_u(\phi) \geq 0$ for all $u \in \mathbf{T}$, and, in particular, $V_T(\phi) > 0$ on A. So, ϕ is type 1 arbitrage.

Solution 2-2

a) We have, for every $m > n$,

$$S_m - S_n = \sum_{k=n+1}^{m} Y_k\beta_k = \sum_{k=n+2}^{m} Y_k\beta_k + Y_{n+1}\beta_{n+1},$$

where

$$\beta_{n+1} > \sum_{k=n+2}^{\infty} \beta_k > \sum_{k=n+2}^{m} \beta_k \geq \sum_{k=n+2}^{m} Y_k\beta_k.$$

We see that $\text{sign}[S_m - S_n] = \text{sign}[Y_{n+1}]$. So, for any $\vartheta = h1_{]\sigma,\tau]}$, where $\sigma < \tau$ and $h \in L^\infty(\mathcal{F}_\sigma)$, we have

$$G_\infty(\vartheta) = h(S_\tau - S_\sigma) > 0 \iff \text{sign}[hY_{n+1}]1_{\{\sigma=n<\tau\}} > 0.$$

On the right hand side, Y_{n+1} is independent of $\mathcal{F}_n$ and takes both values $-1, +1$ with positive probability. We conclude that there does not exist arbitrage for trading strategies of form $h1_{]\sigma,\tau]}$. This means that there does not exist simple arbitrage either. Indeed, if $\sum_{k=1}^{n} h_k 1_{]\tau_{k-1},\tau_k]}$ is an arbitrage opportunity, then there exists a minimal index k such that $V_{\tau_k} \in L^0_+ \setminus \{0\}$. Put $\sigma = \tau_{k-1}$, $\tau = \tau_k$. If $k = 1$ or if $V_{\tau_{k-1}}(\theta) = 0$ we take $h = h^k$. Otherwise, $A := \{V_{\tau_{k-1}}(\theta) < 0\}$ has positive probability and we take $h = h^k 1_A$. Now $h1_{]\sigma,\tau]}$ is an arbitrage opportunity.

b) An equivalent local martingale measure Q must satisfy $Q(Y_n = +1) = Q(Y_n = -1) = \frac{1}{2}$. For the marginals of P and Q, we have $\frac{dP^n}{dQ^n} = (1 + \alpha_n)1_{\{Y_n=+1\}} + (1 - \alpha_n)1_{\{Y_n=-1\}}$ and $\frac{dQ^n}{dP^n} = \frac{1}{1+\alpha_n}1_{\{Y_n=+1\}} + \frac{1}{1-\alpha_n}1_{\{Y_n=-1\}}$. By Kakutani's Dichotomy theorem, we have $P \sim Q$ if and only if

$$\prod_{n=1}^{\infty} \int \Big(\frac{dP^n}{dQ^n}\Big)^{\frac{1}{2}} dQ^n > 0 \text{ and } \prod_{n=1}^{\infty} \int \Big(\frac{dQ^n}{dP^n}\Big)^{\frac{1}{2}} dP^n > 0.$$

Both inequalities yield the same condition:

$$\begin{aligned}
& \prod_{n=1}^{\infty} \Big(\sqrt{\frac{1}{2}(1+\alpha_n)} + \sqrt{\frac{1}{2}(1-\alpha_n)}\Big) > 0 \\
x = \log e^x :\iff & \sum_{n=1}^{\infty} \log\Big(\sqrt{\frac{1}{2}(1+\alpha_n)} + \sqrt{\frac{1}{2}(1-\alpha_n)}\Big) > -\infty \\
e^{-2x} < 1 - x < e^{-x},\ 0 < x < \frac{1}{2} :\iff & \sum_{n=1}^{\infty} \Big(1 - \big(\sqrt{\frac{1}{2}(1+\alpha_n)} + \sqrt{\frac{1}{2}(1-\alpha_n)}\big)\Big) < \infty \\
\iff & \sum_{n=1}^{\infty} \alpha_n^2 < \infty.
\end{aligned}$$

The last equivalence follows from the fact that $(\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{ab} + b$ for $a, b \geq 0$, and

$$\frac{(x-y)^2}{4(1-\delta)} \leq (\sqrt{x} - \sqrt{y})^2 = \Big(\int_x^y \frac{dt}{2\sqrt{t}}\Big)^2 \leq \frac{(x-y)^2}{4\delta}$$

for $0 < \delta < \frac{1}{2}$ and $\delta < x, y < 1 - \delta$.

Solution 2-3

Let us consider a standard Brownian motion W on $[0,T]$, and its (completed) natural filtration $\mathbf{F}=(\mathcal{F}_t)_{t\in[0,T]}$. A normal distribution $N(0,T)$ is non-atomic, so, there exists a function $f:\mathbb{R}\to\mathbb{R}_+$ such that $f(W_T)\sim\mu$. Put $M_t:=E[f(W_T)\mid\mathcal{F}_t]$, $t\in[0,T]$. Then $M_0=E[f(W_T)]=1$ and $M_T=f(W_T)\sim\mu$. The measure μ is fully supported on $\mathbb{R}_+$, so $f\geq 0$, and consequently $M\geq 0$. Because M is a martingale with respect to a Brownian filtration, its paths are continuous almost surely. The distribution of M is a martingale measure on Ω.

Solution 2-4

By Ito formula,

$$dS_t=\sum_i w_i(W_t)dW_t^i+\frac{1}{2}\sum_i w_{i,i}(W_t)dt=:dM_t+\frac{1}{2}\Delta w(W_t)dt=:dM_t+dA_t.$$

Now, if $\Delta w=0$ on $\mathbb{R}^d$, we see that S is a continuous local martingale for every starting point $S_0\in\mathbb{R}^d$. But if we *assume* that S is a continuous local martingale for some $S_0\in\mathbb{R}^d$, then so is $A=S-M$. On the other hand, it is also of bounded variation, so $A=0$. Here, it follows that $\Delta w=0$.

Solution 2-5

The result is known as Strassen's theorem. The necessity follows from Jensen's inequality. We show the sufficiency. We begin by proving an auxiliary result (also from Strassen). We assume the weak topology and finite first moments for probability measures.

Step 1: Lemma. Let us denote $X\times Y:=\mathbb{R}\times\mathbb{R}$, and by π_X and π_Y the projections from $X\times Y$ onto X and Y, respectively. Then the marginals of measure λ on $X\times Y$ can be written as push-forward measures

$$\lambda\circ\pi_X^{-1}\text{ and }\lambda\circ\pi_Y^{-1}.$$

Given a non-empty convex closed set Λ of probability measures on $X\times Y$ and marginals μ and υ, there exists a probability measure $\lambda\in\Lambda$ with $\mu=\lambda\circ\pi_X^{-1}$ and $\upsilon=\lambda\circ\pi_Y^{-1}$ if and only if

$$\int f(x)\mu(dx)+\int g(y)\upsilon(dy)\geq\inf_{\tilde{\lambda}\in\Lambda}\{\int((f\circ\pi_X)(x,y)+(g\circ\pi_Y)(x,y))\tilde{\lambda}(dx\times dy)\},\ \forall f\in C_b(X),\ \forall g\in C_b(Y).$$

The necessity is clear in the statement. We show the sufficiency. Let M_Λ denote the set of all possible pairs of marginals of measures in Λ, i.e.,

$$M_\Lambda:=\{(\tilde{\mu},\tilde{\upsilon}):\exists\tilde{\lambda}\in\Lambda\text{ s.t. }\tilde{\mu}=\tilde{\lambda}\circ\pi_X^{-1}\text{ and }\tilde{\upsilon}=\tilde{\lambda}\circ\pi_Y^{-1}\}.$$

The set M_Λ is convex. Since Λ is closed and projection mappings are continuous, M_Λ is also closed. We have $(\mu,\upsilon)\in M_\Lambda$, as otherwise there would exist $f,g\in C_b(\mathbb{R})$ s.t.

$$\int f(x)\mu(dx)+\int g(y)\upsilon(dy)<\inf_{(\tilde{\mu},\tilde{\upsilon})\in M_\Lambda}\{\int f(x)\tilde{\mu}(dx)+\int g(y)\tilde{\upsilon}(dy)\},$$

which contradicts the assumption.

Step 2: The set of martingale measure satisfies the assumptions of Lemma. The set of martingale measures Λ on $\mathbb{R}^2$ is clearly convex and non-empty. It is also closed:

$$\begin{aligned}\Lambda&:=\{\lambda:\lambda\text{ is a martingale measure on }\mathbb{R}^2\}\\&=\{\lambda:\int y1_A(x)\lambda(dx\times dy)=\int x1_A(x)\lambda(dx\times dy)\ \forall A\in\mathcal{B}(\mathbb{R})\}\\&=\{\lambda:\int yf(x)\lambda(dx\times dy)=\int xf(x)\lambda(dx\times dy)\ \forall f\in C_b(\mathbb{R})\}\\&=\bigcap_{f\in C_b(\mathbb{R})}\{\lambda:\int(y-x)f(x)\lambda(dx\times dy)=0\}.\end{aligned}$$

The first, second and last equality are just definitions and the third follows using standard approximation argument. Assuming finite first moments on the marginals, the mapping

$$\lambda \mapsto \int_{\mathbb{R}^2} (y-x)f(x)\lambda(dx \times dy) = \lim_{K\to\infty} \int_{[-K,K]\times[-K,K]} (y-x)f(x)\lambda(dx \times dy)$$

is continuous, for every $f \in C_b(\mathbb{R})$. So, Λ is an intersection of closed sets: Λ is closed.

Step 3: The set of martingale measures satisfies the condition in Lemma. Given $f, g \in C_b(\mathbb{R})$, let g_0 denote the largest convex minorant of g. We have

$$\begin{aligned}\int f(x)\mu(dx) + \int g(y)\upsilon(dy) &\geq \int f(x)\mu(dx) + \int g_0(y)\upsilon(dy)\\ &\geq \int (f(x) + g_0(x))\mu(dx)\\ &\geq \inf_{x\in\mathbb{R}}\{f(x) + g_0(x)\}.\end{aligned}$$

Let $r \in \mathbb{R}$. We will show

$$r > \inf_{x\in\mathbb{R}}\{f(x) + g_0(x)\} \implies r > \inf_{\tilde{\lambda}\in\Lambda}\{\int ((f\circ\pi_X)(x,y) + g\circ\pi_Y(x,y))\tilde{\lambda}(dx\times dy).$$

The mapping

$$t \mapsto \inf_{\tilde{\upsilon}}\{\int g(y)\tilde{\upsilon}(dy) : \int y\tilde{\upsilon}(dy) = t\}$$

is a convex minorant of g, and, for every $t \in \mathbb{R}$, the linear functional on the right-hand side attains its infimum on the compact domain. So, there exists $s \in \mathbb{R}$ and $\tilde{\upsilon}^*$ s.t.

$$r > f(s)+g_0(s) \geq f(s)+\inf_{\tilde{\upsilon}}\{\int g(y)\tilde{\upsilon}(dy) : \int y\tilde{\upsilon}(dy) = s\} = \int ((f\circ\pi_X)(x,y)+g\circ\pi_Y(x,y))(\delta_s\otimes\tilde{\upsilon}^*)(dx\times dy).$$