ETH Zürich HS 2015
D-MATH
Prof. H. Mete Soner

Coordinator
Matti Kiiski

Mathematical Finance
Solutions Sheet 5

Solution 5-1

a) We first show that $u(t,\cdot)$ is increasing in x. Fix some arbitrary $0 < x_1 \le x_2$ and an admissible control process π. We write $Z_s = X_s^{t,x_2} - X_s^{t,x_1}$. Then the process Z satisfies the SDE

$$dZ_s = Z_s \left[r ds + \pi_s((\mu - r)ds + \sigma dW_s)\right], \quad Z_t = x_2 - x_1 \ge 0.$$

Thus $Z_s \ge 0$ and $X_s^{t,x_2} \ge X_s^{t,x_1}$ for all $s \ge t$. Since U is increasing, we have $U(X_T^{t,x_1}) \le U(X_T^{t,x_2})$ and thus $u(t,x_1) \le u(t,x_2)$.
To show concavity let $0 < x_1, x_2$, $\lambda \in [0,1]$ and π_1, π_2 be two admissible control processes. We write $x_\lambda = \lambda x_1 + (1-\lambda)x_2$. Also X^{t,x_i} is the wealth process starting from x_i at time t and controlled by π_i, where $i \in \{1,2\}$. Set

$$\pi_s^\lambda = \frac{\lambda X_s^{t,x_1}\pi_s^1 + (1-\lambda)X_s^{t,x_2}\pi_s^2}{\lambda X_s^{t,x_1} + (1-\lambda)X_s^{t,x_2}}.$$

By convexity of A, the process π^λ lies in the admissibility class $\mathcal{A}$. Moreover from the linear dynamics of the wealth process, we see that $X^\lambda = \lambda X^{t,x_1} + (1-\lambda)X^{t,x_2}$ is governed by

$$\begin{aligned} dX_s^\lambda &= X_s^\lambda[r ds + \pi_s^\lambda((\mu - r)ds + \sigma dW_s)], \quad s \ge t, \\ X_t^\lambda &= x_\lambda. \end{aligned}$$

Therefore by the concavity of the utility function U,

$$U(\lambda X_T^{t,x_1} + (1-\lambda)X_T^{t,x_2}) \ge \lambda U(X_T^{t,x_1}) + (1-\lambda)U(X_T^{t,x_2}),$$

which implies that

$$u(\lambda x_1 + (1-\lambda)x_2) \ge \lambda U(X_T^{t,x_1}) + (1-\lambda)U(X_T^{t,x_2}).$$

Since π^1 and π^2 are arbitrary, we conclude that

$$u(\lambda x_1 + (1-\lambda)x_2) \ge \lambda u(x_1) + (1-\lambda)u(x_2).$$

b) The dynamic programming principle is given as

$$u(t,x) = \sup_{\pi \in \mathcal{A}} E\left[u(\theta, X_\theta^{t,x,\pi}) \middle| \mathcal{F}_t \right],$$

for $\theta \in [t,T]$ and the dynamic programming equation is

$$-u_t(t,x) - \sup_{\pi \in A}\left\{ (xr + \pi(\mu - r)x)u_x(t,x) + \frac{1}{2}x^2\pi^2\sigma^2 u_{xx}(t,x)\right\} = 0$$

with the boundary condition $u(T,x) = U(x)$.

Solution 5-2

We are looking for a candidate solution of the form

$$w(t,x) = \phi(t)U(x),$$

for some positive function $\phi(t)$. Then observing $xU'(x) = \gamma U(x)$ and $x^2U''(x) = \gamma(\gamma-1)U(x)$, we get

$$\phi'(t) + \gamma \sup_{\pi\in A}\left(r + \pi(\mu - r) + \frac{1}{2}\pi^2\sigma^2(\gamma-1)\right)\phi(t) = 0$$

and $\phi(T) = 1$. Then the candidate optimal control $\hat{\pi}$ is

$$\hat{\pi} = \arg\max_{\pi\in A}\left\{r + \pi(\mu - r) + \frac{1}{2}\pi^2\sigma^2(\gamma-1)\right\} = \frac{\mu - r}{\sigma^2(1-\gamma)}$$

so that

$$\gamma \sup_{\pi\in A}\left(r + \pi(\mu - r) + \frac{1}{2}\pi^2\sigma^2(\gamma-1)\right) = \frac{(\mu-r)^2}{2\sigma^2}\frac{\gamma}{1-\gamma} + r\gamma =: \rho.$$

Hence $\phi(t) = \exp(\rho(T-t))$ and the candidate solution is $w(t,x) = \exp(\rho(T-t))U(x)$.

Solution 5-3

Since $w \in C^{1,2}([0,T]\times\mathbb{R})$, we have for all $(t,x)\in[0,T]\times\mathbb{R}$, $\pi\in\mathcal{A}$, $s\in[t,T]$ and any stopping time τ valued in $[t,\infty)$, by Ito's formula

$$\begin{aligned} w(s\wedge\tau, X^{t,x}_{s\wedge\tau}) &= w(t,x) + \int_t^{s\wedge\tau} w_x(\tilde{t}, X^{t,x}_{\tilde{t}})X^{t,x}_{\tilde{t}}\pi_{\tilde{t}}\sigma dW_{\tilde{t}} \\ &+ \int_t^{s\wedge\tau} w_t(\tilde{t}, X^{t,x}_{\tilde{t}}) + (r + \pi_{\tilde{t}}(\mu - r))X^{t,x}_{\tilde{t}} w_x(\tilde{t}, X^{t,x}_{\tilde{t}}) + \frac{1}{2}\sigma^2\pi^2_{\tilde{t}}[X^{t,x}_{\tilde{t}}]^2 w_{xx}(\tilde{t}, X^{t,x}_{\tilde{t}})d\tilde{t}. \end{aligned}$$

We choose

$$\tau_n = \inf\left\{s \geq t : \int_t^s |w_x(\tilde{t}, X^{t,x}_{\tilde{t}})X^{t,x}_{\tilde{t}}\pi_{\tilde{t}}|^2 d\tilde{t} \geq n\right\}$$

and observe that $\tau_n \uparrow \infty$. The stopped process $\{\int_t^{s\wedge\tau_n} w_x(\tilde{t}, X^{t,x}_{\tilde{t}})\pi_{\tilde{t}}X^{t,x}_{\tilde{t}}\sigma dW_{\tilde{t}}, \quad t \leq s \leq T\}$ is then a martingale and by taking expectations we get

$$\begin{aligned} &E[w(s\wedge\tau_n, X^{t,x}_{s\wedge\tau_n})] \\ &\quad = w(t,x) + E\left[\int_t^{s\wedge\tau} w_t(\tilde{t}, X^{t,x}_{\tilde{t}}) + (r + \pi_{\tilde{t}}(\mu - r))X^{t,x}_{\tilde{t}} w_x(\tilde{t}, X^{t,x}_{\tilde{t}}) + \frac{1}{2}\sigma^2\pi^2_{\tilde{t}}[X^{t,x}_{\tilde{t}}]^2 w_{xx}(\tilde{t}, X^{t,x}_{\tilde{t}})d\tilde{t}\right] \end{aligned}$$

Since w satisfies the HJB equation, we have for all $\pi\in\mathcal{A}$ that

$$\int_t^{s\wedge\tau} w_t(\tilde{t}, X^{t,x}_{\tilde{t}}) + (r + \pi_{\tilde{t}}(\mu - r))X^{t,x}_{\tilde{t}} w_x(\tilde{t}, X^{t,x}_{\tilde{t}}) + \frac{1}{2}\sigma^2\pi^2_{\tilde{t}}[X^{t,x}_{\tilde{t}}]^2 w_{xx}(\tilde{t}, X^{t,x}_{\tilde{t}})d\tilde{t} \leq 0,$$

and so

$$E[w(s\wedge\tau_n, X^{t,x}_{s\wedge\tau_n})] \leq w(t,x).$$

Observe that for all $t\in[0,T]$,

$$|w(t,x)| = \left|\exp(\rho(T-t))\frac{x^\gamma}{\gamma}\right| \leq \frac{\exp(\rho T)}{\gamma}(1+x)^\gamma \leq \frac{\exp(\rho T)}{\gamma}(1+x)^2 \leq 2\frac{\exp(\rho T)}{\gamma}(1+x^2),$$

since $\gamma < 1$. Therefore,

$$|w(s \wedge \tau_n, X_{s\wedge\tau_n})| \le C \left(1 + \sup_{s\in[t,T]} |X_s^{t,x}|^2\right).$$

By assumption, the right-hand-side term is integrable. By applying the dominated convergence theorem, as $n \to \infty$ we obtain

$$E\left[w(s, X_s^{t,x})\right] \le w(t,x)$$

By continuity of w on $[0,T]\times\mathbb{R}$, by sending s to T, we obtain by dominated convergence theorem that

$$E\left[U(X_T^{t,x})\right] \le w(t,x)$$

for any $\pi \in \mathcal{A}$ so that $u(t,x) \le w(t,x)$.
To have the converse inequality, we apply Ito to w, but this time we use the candidate optimal control $\hat{\pi}$. Since

$$\begin{aligned}
&- w_t(t,x) - \sup_{\pi\in A}\left\{(xr + \pi(\mu - r)x)w_x(t,x) + \frac{1}{2}x^2\pi^2\sigma^2 w_{xx}(t,x)\right\} \\
&= -w_t(t,x) - (xr + \hat{\pi}(\mu - r)x)w_x(t,x) - \frac{1}{2}x^2\hat{\pi}^2\sigma^2 w_{xx}(t,x) = 0,
\end{aligned}$$

following the same steps as above we obtain

$$w(t,x) = E\left[w(s, \hat{X}_s^{t,x})\right] \le u(t,x),$$

where $\hat{X}$ represents the solution of the SDE

$$d\hat{X}_s = \hat{X}_s\left(rds + \hat{\pi}_s((\mu - r)ds + \sigma dW_s)\right)$$

with the control $\hat{\pi}$. Thus $w(t,x) = u(t,x) = E\left[w(s, \hat{X}_s^{t,x})\right]$. We conclude that w is the value function u and $\hat{\pi}$ is the optimal control.