Mathematical Finance

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This version: January 8, 2012

Contents

Chapter 1

Basics of Financial Markets

Lecture 2, September 26, 2011

The basic goal of this chapter is to develop the mathematical structure needed to study the financial markets.

We assume that we are given the following objects.

- The probabilistic structure is described through a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where as usual $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ is a filtration with usual conditions. We always think that $\mathcal{F}$ describes information or events observable up to time t.

- Maturity or a time horizon $T \in (0, \infty)$. In continuous time models all times are accepted as trading dates $t \in [0, T]$. While, in discrete time models, only a finite subset is accepted. The discrete models have the advantage of a simpler mathematical structure. However, continuous models have a richer structure and more developed mathematical tools.

  Note that one can naturally embed discrete time models into continuous ones by simply taking $\mathbb{F}$ and all processes piecewise constant.

- One asset is the bank account. We always use it as the denomination basis. Hence its price is given by $B_t = 1$, for all t.

- There are d risky assets with price processes $S^i = (S^i_t)_{0 \le t \le T}$, in units of the bank account. Clearly, S is a $\mathbb{R}^d$-valued stochastic process. S^i_t is the price of the asset i at time t, so S must be at least adapted to $\mathbb{F}$.

The simplest example of the above structure is a discrete time model.

Example 1.1 Binomial model of Cox-Ross-Rubinstein. This is a discrete time model with $\tilde{B}_k = (1 + r)^k$ and $\tilde{S}_{k+1}/\tilde{S}_k$ has identically and independently distributed values $1 + u$, $1 + d$ with probabilities p and $1 - p$. ∎

The second example coves almost all continuous time models. But the one dimensional constant coefficient model is known as the Black & Scholes model.

Example 1.2 Black Scholes Model. We assume that the bank account has a constant interest rate r, so $\tilde{B}_t = e^{rt}$. There is only one stock and its price follows a geometric Brownian motion (GBM) with constant *mean return rate* μ and *volatility* σ. Then,

$$\tilde{S}_t = S_0 \ exp(\sigma W_t + (\mu - \frac{\sigma^2}{2})t)$$

where W is a one dimensional Brownian motion. By discounting, $B_t := (\tilde{B}_t/\tilde{B}_t) \equiv 1$ and

$$S_t := \frac{\tilde{S}_t}{\tilde{B}_t} = S_t = S_0 \ exp(\sigma W_t + (\mu - r - \frac{\sigma^2}{2})t).$$

By an easy application of the Itô formula,

$$dS_t = S_t \left((\mu - r)dt + \sigma dW_t\right).$$

More generally, one may utilize the *Itô* process model ,

$$dS_t^i = S_t^i \left(b_t^i dt + \sum_{j=1}^{n} \sigma_t^{ij} dW_t^i \right)$$

with predictable processes $b \in \mathbb{R}^d$ and $\sigma \in \mathbb{R}^{d \times n}$. Models with non-deterministic coefficients are known as *stochastic volatility* models. However, this generality may not be tractable. Then, one may impose further structure on these processes.

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We continue with a description of several important processes that will be repeatedly used in our analysis.

- Trading strategy or dynamic portfolio is an adapted (sometimes predictable) stochastic process $\varphi = (\varphi_t)_{0 \le t \le T}$, with two components $\varphi = (\vartheta, \eta)$. This process is chosen by the investor and has the following interpretation. The real-valued process η_t is the number of units of the bank account B held by the investor and ϑ is an $\mathbb{R}^d$ process whose i-th component ϑ_t^i is the number of shares/units of asset i held at time t.

- The value process $V(\varphi) = (V_t(\varphi))_{0 \le t \le T}$ is the marked-to-market value of the portfolio at time t. Hence,

  $$V_t(\varphi) = \Sigma_{i=1}^d \vartheta_t^i S_t^i + \eta_t 1 = \vartheta \cdot S_t + \eta_t.$$

  In financial markets with friction, this value may not be instantaneously achieved in cash due to liquidity, transaction cost or tax reasons. These markets will be considered only later in these notes.

  We also remark that for the ease of notation, we mostly think of one risky asset (i.e., $d = 1$) and usually omit scalar products etc. (This is usually harmless.)

- Cost of a strategy. We motivate this definition through a piece-wise constant strategies. Indeed, suppose that we keep a strategy φ constant between t and $t + \Delta t$ and only change it from φ_t to $\varphi_{t+\Delta t}$ at time t. Then in the interval $(t, t+\Delta t]$ the cost of this trading strategy is given by

$$\begin{aligned} C_{t+\Delta t} - C_t &= (\varphi_{t+\Delta t} - \varphi_t) \cdot (S_t, B_t) \\ &= \vartheta_{t+\Delta t} S_t + \eta_{t+\Delta t} - \vartheta_t S_t - \eta_t - \vartheta_{t+\Delta t} S_{t+\Delta t} + \vartheta_{t+\Delta t} S_{t+\Delta t} \\ &= V_{t+\Delta t} - V_t - \vartheta_{t+\Delta t}(S_{t+\Delta t} - S_t). \end{aligned}$$

Summing up and taking Δt small suggests the following as the natural definition for cumulative cost process,

$$C_t(\varphi) := V_t(\varphi) - \int_0^t \vartheta_u dS_u, \quad 0 \leq t \leq T.$$

Clearly, the above definition requires the stochastic integral to be well defined. In view of general results in stochastic processes S must be a semi-martingale and ϑ must be predictable in order to define this integral. See the Remark 1.3 below for a discussion of this.

$C_t(\varphi)$ is defined to be the total cost on $[0, t]$ from trading according to φ.

We continue with a short remark about the measurability issues.

Remark 1.3 In discrete time, S is piecewise constant and the above integral reduces to a sum. Only condition on φ is that stock holdings ϑ_t on $(t, t+\Delta t]$ must be determined at begining at time t, to exclude insider or prophetic knowledge about S. This means that ϑ must be predictable with respect to $\mathbb{F}$. However, we can adjust bank account at the end $t + \Delta t$; so η is adapted to $\mathbb{F}$. This is asymmetric because S is risky, while B is riskless.

In continuous time, we still impose that ϑ is predictable and η is adapted. In addition we want S to be semimartingale and ϑ to be S-integrable so that stochastic integral $\int \vartheta dS$ is well defined and again semimartingale. (At least if S is continuous, we know sufficient conditions for S-integrability.)

However, when S is a continuous, Ito process as in Example 1.2 and the filtration is generated by this process, then one may get away with adapted strategies ϑ.

Finally, we remark that $V(\varphi)$, $C(\varphi)$ and $\int \vartheta dS$ are always $\mathbb{R}$-valued. If ϑ and S are $\mathbb{R}^d$-valued, $\int \vartheta dS$ denotes vector stochastic integration, which may differ from

$$\sum_{i=1}^{d} \int \vartheta^i dS^i.$$

This difference can cause technical problems.

Definition 1.4 A strategy $\varphi = (\vartheta, \eta)$ is called *self-financing* if

$$C(\varphi) \equiv C_0(\varphi), \quad i.e. \quad C_t(\varphi) = C_0(\varphi) \quad P - a.s., \forall t$$

Hence a self-financing strategy φ, after initial outlay of $C_0(\varphi) = V_0(\varphi)$ to set up strategy, trading generates neither expenses nor surplus. Allowing any self- financing strategy in a model is not a good idea. In fact, later we shall need additional restrictions.

Lemma 1.5 *A strategy $\varphi = (\vartheta, \eta)$ is self-financing if and only if*

$$V(\varphi) = V_0(\varphi) + \int \vartheta dS.$$

Indeed, there exists a bijection between self-financing strategies $\varphi = (\vartheta, \eta)$ and the pairs (V_0, ϑ), where $V_0 \in L^0(\mathcal{F}_0)$ and ϑ is predictable and S-integrable. Explicitly, $V_0 = V_0(\varphi)$ and

$$\eta = V_0 + \textstyle\int \vartheta dS - \vartheta \cdot S. \tag{1.0.1}$$

Moreover, if $\varphi = (\vartheta, \eta)$ is self-financing, then η is also predictable.

Proof. We observe that

1. $C(\varphi) = V(\varphi) - \int \vartheta dS$
2. $\eta = V(\varphi) - \vartheta \cdot S.$

We now combine the two to arrive at (1.0.1).

To prove the predictability of η, we recall that for RCLL process (a right continuous process with left limits) $Y = (Y_t)_{0 \leq t \leq T}$, $\Delta Y_t := Y_t - Y_{t-}$ denotes jump at time t. From the stochastic integration theory,

$$\Delta \left(\textstyle\int \vartheta dS\right)_t = \vartheta_t \cdot \Delta S_t = \vartheta_t \cdot S_t - \vartheta_t \cdot S_{t-}.$$

By (1.0.1),

$$\begin{aligned}\eta_t &= V_0 + \textstyle\int_0^t \vartheta_u dS_u - \vartheta_t \cdot S_t \\ &= V_0 + \textstyle\int_0^{t-} \vartheta_u dS_u + \Delta \left(\int \vartheta dS\right)_t - \vartheta_t \cdot S_t \\ &= V_0 + \textstyle\int_0^{t-} \vartheta_u dS_u - \vartheta_t \cdot S_{t-}.\end{aligned}$$

The second term is adapted and locally continuous, hence is predictable. In the third term (S_{t-}) is predictable by the same argument. Finallyϑ by assumption.

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Exercise: Do parts of Lemma refl.self-financing more explicitly in discrete time.

Remark 1.6 $G(\vartheta) = \int \vartheta dS = 0 + \int \vartheta dS$ is by L1.1 value process of self-financing strategy with initial capital $V_0 = 0$ and trading via ϑ; cumulative gains/losses(depending on sign) from ϑ.

Important implicit assumptions in our model setup are the following:

- One may trade continuously in time;
- Prices for buying and selling shares are both given by S. Hence there are no transaction costs and trading is frictionless
- ϑ is $\mathbb{R}^d$-valued. In other words, ${\vartheta_t}^i$ can take arbitrary and even negative values. This means no trading constraints (like e.g minimal lot size or integer number of units). In particular, short sales (${\vartheta_t}^i < 0$) and borrowing ($\eta_t < 0$) are allowed.

- Asset prices are exogenously given by fixed process S, do not react to trading strategies. This means that our agents are small investors or price takers. As a result, the book value $V(\varphi)$ is also a reasonable as market / liquidation value.

Example 1.7 Take $d = 1$ and let $S = W$ be a standard Brownian Motion. For simplicity, work on $[0, \infty]$; (One could use time change to get to $[0, T]$). The stopping time $\tau := \inf\{t \geq 0 \mid W_t = 1\}$ has $\tau < \infty$ $\mathbb{P}$-a.s.; so $\vartheta := I_{(0,\tau]}$ is predictable. Then, the self-financing strategy with $V_0 = 0$ and ϑ is given by

$$G_\infty(\vartheta) = \int_0^\infty \vartheta_u dW_u = W_\tau - W_0 = 1.$$

So we start with zero initial capital and end up without intermediate surplus or expenses, with final wealth 1. This is a money pump!

One problem with this strategy is its value process

$$V_t((0, \vartheta)) = \int_0^t \vartheta_u dS_u = W_{t\wedge\tau} = W_t{}^\tau$$

is unbounded from below. In other words, before ending up at 1, we might have to borrow huge amounts of money!

If $W^\tau \geq -a$, then the martingale W^τ is a supermartingale by Fatou's Lemma and bounded below. Hence it is closable from the right and we can apply stopping theorem to conclude that

$$\mathbb{E}[W_\tau] = \mathbb{E}[W_\infty^\tau] \leq \mathbb{E}[W_0^\tau] = 0, \tag{1.0.2}$$

which is false since $W_\tau = 1$ $\mathbb{P}$-a.s.

It is not important that $S = W$ becomes negative; we can construct similar example when S is a geometric Brownian motion.

With above setup, we can now formulate two central problems of hedging and of optimal investment:

1. Given $H \in L^0(\mathcal{F}_T)$ – a random payoff at time T – can we find a self-financing strategy (V_0, ϑ) such that $V_T(\varphi) = H$, $\mathbb{P}$-a.s. (or perhaps $V_T(\varphi) \geq H$, $\mathbb{P}$-a.s.)? If yes, what is (minimal) required initial capital V_0?

2. Given an initial capital x, what is best investment strategy, i.e., which self-financing strategy (x, ϑ) produces the "best" final wealth $V_t(\varphi) = x + \int_0^T \vartheta_u dS_u$? Clearly requires (subjective) criterion to compare different final wealths.

Chapter 2

Arbitrage and martingale measures

Lecture 3, September 29, 2011

In a reasonable model for a financial market, we should not have a way of making money from nothing. The goal of this section is to formalize this statement. So we need

(a) first to define "making money from nothing" or the notion of *no-arbitrage* mathematically,

(b) and then obtain necessary and sufficient conditions for this not to happen.

As always our ground model consists of a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ over $[0, T]$ with $B \equiv 1$ and S is adapted to $\mathbb{F}$ with values in $\mathbb{R}^d$. We also have a strategy, ϑ, is such that

i . ϑ is $\mathbb{F}$-predictable,

ii . $\int \vartheta dS$ is well-defined.

Note that we have not made any assumptions on S yet. So the second assertion is very hard and imprecise! We also assume that ϑ is self-financing strategy. In view of Chapter 1, this is equivalent to

$$\text{Gain-Loss process} \quad G_t(\vartheta) = \int_0^t \vartheta_u dS_u \quad \forall\ 0 \le t \le T.$$

In view of the Example 1.7 of Chapter 1, we impose an *admissibility condition* on this strategy as well. Namely, there exists a constant $a \in \mathbb{R}$ so that

$$G_t(\vartheta) \ge -a \quad \forall\ 0 \le t \le T, \quad \mathbb{P} - a.s.$$

It is important to note that a may depend on ϑ. We also note that there are weaker conditions, allowing random lower bound as well.

Definition 2.1 Let Θ_{adm} be the set of all admissible strategies.

To summarize ϑ is a

$$\begin{aligned} \text{strategy} &\Leftrightarrow \mathbb{R}^d\text{-valued, predictable, } S\text{-integrable} \\ \text{admissible} &\Leftrightarrow G_T(\vartheta) \ge -a \end{aligned}$$

One may now ask what kind of portfolio processes are admissible. Essentially, there are two classes of examples. First one is the class of simple strategies. For these the integration is defined in an elementary way.

Indeed, we say that ϑ is a simple-strategy, denoted by $b\varepsilon$, if

$$\vartheta_u = \sum_{i=1}^{n} h^i \chi_{(\tau_{i-1},\tau_i]}(u)$$

where $n \in \mathbb{N}$ deterministic, $0 \leq \tau_0 < ... < \tau_n = T$ $\mathbb{F}$-stopping times, for each i $h^i \in L^\infty(\mathcal{F}_{\tau_{i-1}}, \mathbb{P}, \mathbb{R}^d)$. Notice that for $\vartheta \in b\varepsilon$

$$\int_0^t \vartheta_u dS_u = \sum_{i=1}^{n} h^i (S_{\tau_i \wedge t} - S_{\tau_{i-1} \wedge t}).$$

Hence the stochastic integral is defined. In a discrete time model

$$b\varepsilon = \{\text{all bounded, } \mathbb{R}^d\text{-valued, predictable } \vartheta' s\}.$$

In the second case, we assume that the stock process S is a semi-martingale. Then, one can define $\int \vartheta_u dS_u$ for a large class of integrands. Moreover, we have a well-developed theory of integration.

2.1 No-arbitrage conditions

A simple arbitrage opportunity is a strategy $\vartheta \in b\varepsilon \cap \Theta_{adm}$ with a non-negative final gains $G_T(\vartheta) \geq 0$ which is strictly positive with positive probability. We may write this as

$$G_T(\vartheta) \geq 0, \mathbb{P} - a.s. \text{ and } \quad \mathbb{P}(G_T(\vartheta) > 0) > 0 \quad \Leftrightarrow \quad G_T(\vartheta) \in \mathcal{L}^0_+(\mathcal{F}_T) \backslash \{0\},$$

where for a given σ-algebra $\mathcal{G}$, $\mathcal{L}^0(\mathcal{G})$ is the set of all $\mathcal{G}$ measurable real-valued random variables which are finite $\mathbb{P}$ almost surely and $\mathcal{L}^0_+(\mathcal{G})$ is the set of all $\mathbb{P}$ almost surely non-negative elements in $\mathcal{L}^0(\mathcal{G})$.

We are now in a position to define no-arbitrage precisely.

Definition 2.2 We say that a financial market satisfies the *no-arbitrage condition with elementary strategies* and abbreviate it by (NA^{adm}_{elem}) if

$$G_T(b\varepsilon^{adm}) \cap \mathcal{L}^0_+(\mathcal{F}_T) = \{0\}.$$

For a semimartingale S, we say that a financial market satisfies the *no-arbitrage condition* and abbreviate it by (NA) if

$$G_T(\Theta_{adm}) \cap \mathcal{L}^0_+(\mathcal{F}_T) = \{0\}.$$

In the definition of no-arbitrage, the choice of admissible strategies is very important. In particular, he lower bound we impose is certainly important both in continuous and infinite discrete time. We give the following example to illustrate this point.

Example 2.3 (Doubling strategy) Consider an infinite discrete time model. Then, $\{S_k\}_{k=1}^{\infty}$ is a sequence of random variables.

Assume that there are $\epsilon > 0$ and $\delta > 0$ so that

$$\mathbb{P}(S_{k+1} \geq S_k + \epsilon | \mathcal{F}_k) \geq \delta, \quad \text{a.s.}$$

Choose a self-financing strategy $\vartheta = (\vartheta_R)$ given by $\vartheta_0 = 1$. Hence, $\eta_0 = -S_0$ and $G_0(\vartheta) = 0$.

If $S_1 \geq S_0 + \epsilon$, then we set $\vartheta_k = 0$ for all $k \geq 1$. This yields

$$\eta_k \geq \epsilon \quad \text{and} \quad G_k(\vartheta) \geq \epsilon > 0.$$

However, if $S_1 \ngeq S_0 + \epsilon$, then we choose $\vartheta_1 = 1 - (S1 - S_0)/\epsilon$. Then,

$$G_1(\vartheta) = (S_1 - S_0) = V_1(\vartheta).$$

Now if $S_2 \geq S_1 + \epsilon$, then again we set $\vartheta_k = 0$ for all $k \geq 2$. This yields

$$G_k = G_2(\vartheta) = (S_1 - S_0) + \vartheta_1(S_2 - S_1) \geq \vartheta_1\epsilon + (S_1 - S_0) = \epsilon.$$

Also note that $G_2 < \epsilon$ in other cases

Let τ be the stopping time given by

$$\tau := \inf\{k : S_k \geq S_{k-1} + \epsilon \quad \text{and} \quad S_j < S_{j-1} + \epsilon, \forall j < k\}.$$

Choose $\vartheta_k = 1 - G_k(\vartheta)/\epsilon$ until τ recursively so that on the event $\{\tau = k+1\}$

$$G_{k+1} = \vartheta_k(S_{k+1} - S_k) + G_k(\vartheta) \geq \vartheta_k\epsilon + G_k(\vartheta) \geq \epsilon.$$

For $\tau < k$ we set $\vartheta_k \equiv 0$. Hence,

$$G_k(\vartheta) = G_\tau(\vartheta) \geq \epsilon.$$

Moreover, $\mathbb{P}(\tau < \infty) = 1$. Hence this is arbitrage but there may not exist a uniform lower bound. Namely,

$$\nexists a \in \mathbb{R}^1 \text{ so that } G_k(\vartheta) \geq -a \quad \forall k!$$

■

We have the following sufficient condition.

Lemma 2.4 (Sufficient Condition) *Suppose there exists $\mathbb{Q} \approx \mathbb{P}$ such that S is a local $\mathbb{Q}$-martingale, then both (NA) and (NA^{adm}_{elem}) both hold.*

Proof. Since $S \in M_{loc}(\mathbb{Q})$ and $\mathbb{Q} \approx \mathbb{P}$, then by Girsanov Theorem S is semimartingale. (This is a simple case of the general Girsanov Theorem but one needs the full power of the theorem). Also it is clear that it suffices to prove (NA). Our goal is to show that

$$\mathbb{E}^{\mathbb{Q}}[G_T(\vartheta)] \leq 0 \quad \forall \vartheta \in \Theta_{adm}. \tag{2.1.1}$$

If this holds, we conclude that for any $\vartheta \in \Theta_{adm}$ with $G_T(\vartheta) \geq 0$, $\mathbb{P}$-a.s., we also have $G_T(\vartheta) \geq 0$, $\mathbb{Q}$-a.s.. Together with (2.1.1) this implies that $G_T(\vartheta) = 0$, $\mathbb{Q}$-a.s., and consequently $G_T(\vartheta) = 0$, $\mathbb{P}$-a.s. This proves (NA).

We know that for $\vartheta \in \Theta_{adm}$, $G_t(\vartheta) = \int_0^t \vartheta_u S_u$ is well-defined and $G_t(\vartheta) \geq -a$ for all $t \in [0,T]$. Then, by the Ansel-Stricker theorem, $G(\vartheta) \in M_{loc}(\mathbb{Q})$.

Since $G(\vartheta) \in M_{loc}(\mathbb{Q})$ and $G(\vartheta) \geq -a$, by Fatou's lemma $G(\vartheta)$ is a $\mathbb{Q}$ super-martingale,

$$\mathbb{E}^{\mathbb{Q}}[G_T(\vartheta)] \leq \mathbb{E}^{\mathbb{Q}}[G_0(\vartheta)] = 0$$

■

Definition 2.5 For a given stock price process S on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and an *equivalent (local) martingale measure* denoted by E(L)MM is a probability measure $\mathbb{Q}$ on the same filtered probability space satisfying

i. $\mathbb{Q} \approx \mathbb{P}$

ii. S is a $\mathbb{Q}$ (local)- martingale.

We denote by $\mathcal{P}^e$ the set of all EMM's and by $\mathcal{P}^e_{loc}$ the set of all ELMM's.

Restatement of Lemma 2.1:

$$\mathcal{P}^e_{loc} \neq \emptyset \quad \Rightarrow \quad (NA).$$

The important question is the converse. We quickly summarize several important facts along this direction. facts.

1. Finite discrete time models are special and the converse is correct. We will prove this in the chapter.

2. In general, the converse is not true. An example in infinite discrete time is given below in Example 2.6.

3. To obtain a general result, one needs to strengthen the "no-arbitrage" conditions. Indeed, as we have seen doubling strategies are key arbitrage constructions since these constructions require infinite trading, we also need to exclude "bad limits" in our definition of no-arbitrage. This goes under name "no-free-lunch-with-vanishing-risk" (NFLVR). We refer to the papers and the recent book by F. Delbaen and W. Schachermayer.

Example 2.6 (Counter-example in infinite discrete time) This is an example of a financial market which has the the property (NA) but there is no $ELMM$.

The stock price process is generated by the recursive equation

$$S_n = S_{n-1} + \beta_n Y_n,$$

where $\{\beta_n\}$'s are deterministic numbers which we choose as $\beta_n = 3^{-n}$. $\{Y_n\}$'s are $\mathbb{P}$-independent sequence with values in $\{-1, +1\}$ and

$$\mathbb{P}(Y_n = +1) = \frac{1}{2}(1 + \alpha_n)$$

and again $\{\alpha_n\}$'s are deterministic and to be chosen.

$$\mathcal{F} = \mathcal{F}^Y = \mathcal{F}^S.$$

Then an ELMM $\mathbb{Q}$ must satisfy $\mathbb{Q}(Y_n = +1) = 1/2$. But by Williams 14.17 or by Stromberg pages 192-193,

$$\mathbb{Q} \approx \mathbb{P} \;\Leftrightarrow\; \sum_{n=1}^{\infty} \alpha_n^2 < \infty$$

So to work for a counterexample, we simply take $\{\alpha_n\}$'s so that

$$\sum \alpha_n^2 = \infty \;\Rightarrow\; Q \perp P.$$

Then, there exist no ELMM for S.

We now proceed to prove that this model with appropriately chosen $\{\beta_n\}$'s. Indeed,

$$\beta_n = 3^{-n} \Rightarrow \beta_n > \sum_{k=n+1}^{\infty} \beta_k.$$

Then for any $m > n$

$$S_m - S_n = (\sum_{k=n+1}^{m} Y_k \beta_k) = \sum_{k=n+2}^{m} Y_k \beta_k + \beta_{n+1} Y_{n+1}$$

If $Y_{n+1} = 1$, then

$$\beta_{n+1} Y_{n+1} = \beta_{n+1} > \sum_{k=n+2}^{\infty} \beta_k \geq \sum_{k=n+2}^{m} \beta_k > \sum_{k=n+2}^{m} Y_k \beta_k$$

A similar computation when $Y_{n+1} = -1$ yields that

$$\mathrm{sign}[S_m - S_n] = \mathrm{sign} Y_{n+1}, \quad \forall m > n.$$

For $\vartheta = h\chi_{(\sigma,\tau]}$ with σ, τ $\mathbb{F}$-stopping times,

$$G_\infty(\vartheta) = h[S_\tau - S_\sigma]$$

and

$$G_\infty(\vartheta) \geq 0 \;\Leftrightarrow\; \mathrm{sign}(hY_{n+1})\chi_{\{A_n\}} > 0,$$

where $A_n := \{\sigma = n < \tau\} \in \mathcal{F}_n$. But since Y_{n+1} is independent of $\mathcal{F}_n$ with values in $\{\pm 1\}$, above is not possible. Hence we can not achieve arbitrage by trading strategies of the form $h\chi_{(\sigma,\tau]}$. But this is equivalent to (NA^{adm}_{elem}). (c.f., Delbaen & Schachermayer).

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Lecture 4, October 3, 2011

2.2 One step Model

A very good reference for this material is the Chapter 1 in Föllmer & Schied.
We assume that there $d+1$ assets $(B, S) \in \mathbb{R}^1 \times \mathbb{R}^d$ as before.

$$S_1(\omega) = (S_1^1(\omega), ..., S_1^d(\omega))$$

is the price of the assets if scenario $\omega-$occurs. This is a very general set-up and we do not even assume $S \in \mathcal{L}^1(\mathbb{P}, \mathcal{F})$.

In this simple setting a portfolio is a deterministic vector $(\eta, \vartheta) \in \mathbb{R}^1 \times \mathbb{R}^d$. At time $t = 1$ the value of the portfolio is given by

$$V_1(\eta, \vartheta) = \eta + \vartheta \cdot S_1,$$

which is a $\mathcal{F}$-mbl random variable. However, the cost of a portfolio is the deterministic function

$$C(\eta, \vartheta) = V_0(\eta, \vartheta) = \eta + \vartheta \cdot S_0$$

Self-financing in this context means there is no endowment put into the system at time $t = 1$. Then a self-financing with zero initial cost means

$$\eta = -\vartheta \cdot S_0$$

and

$$V_1(\eta, \vartheta) := v(\vartheta) = -\vartheta \cdot S_0 + \vartheta \cdot S_1 = \vartheta \cdot (S_1 - S_0) = G(\vartheta).$$

The notion of arbitrage also simplifies. Indeed, an arbitrage opportunity is a deterministic vector, $\vartheta \in \mathbb{R}^d$ satisfying

$$\vartheta \cdot (S_1(\omega) - S_0) \geq 0 \quad \mathbb{P}\text{-almost every } \omega,$$

and

$$\mathbb{P}(\vartheta \cdot (S_1 - S_0) > 0) > 0.$$

Finally, $\mathbb{Q}$ is an equivalent martingale measure if $\mathbb{Q} \approx \mathbb{P}$, $S \in \mathcal{L}^1(\mathbb{Q}, \mathcal{F})$ and

$$\mathbb{E}_{\mathbb{Q}}(S_1^i) = S_0^i \quad \forall i = 1, ..., d.$$

Notice that today's price (or value) of any asset is simply obtained by averaging its future values. Because of this $\mathbb{Q}$ is interpreted as a *pricing operator* and $d\mathbb{Q}/d\mathbb{P}$ as a *pricing kernel*.

In this simple market, we have the following result.

Theorem 2.7 *Above one-step market is arbitrage free if and only if there exists a EMM.*

Proof. Sufficiency. Let $\mathbb{Q}$ be an EMM. Then for any $\vartheta \in \mathbb{R}^d$

$$\mathbb{E}^{\mathbb{Q}}(\vartheta \cdot (S_1 - S_0)) = 0.$$

If $\vartheta \cdot (S_1 - S_0)$ is $\mathbb{P}$-almost surely non-negative, it is so under $\mathbb{Q}$ as well. Then we have

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}(\vartheta \cdot (S_1 - S_0)) = 0, \quad \text{and} \quad & \vartheta \cdot (S_1 - S_0) \geq 0 \\ & \Rightarrow \mathbb{Q}(\vartheta \cdot (S_1 - S_0) \geq 0) = 0 \\ & \Rightarrow \mathbb{P}((\vartheta \cdot (S_1 - S_0) \geq 0) = 0. \end{aligned}$$

So there is no arbitrage opportunity $\vartheta \in \mathbb{R}^d$.

Necessity. Set $Y := S_1 - S_0$. We first assume that

$$Y \in \mathcal{L}^1(\mathbb{P}, \mathcal{F}) \Leftrightarrow \mathbb{E}^{\mathbb{P}}(|Y|) < \infty.$$

We now define a convex set of measures by

$$\mathscr{O} := \{\mathbb{Q} \text{ is a probability measure on } (\Omega, \mathcal{F}),\ \mathbb{Q} \approx \mathbb{P} \text{ and } d\mathbb{Q}/d\mathbb{P} \text{ is bounded}\}.$$

Then, $\mathscr{O}$ has two important properties,

i. $\mathscr{O}$ is convex;

ii. For any $\mathbb{Q} \in \mathscr{O}$, $Y \in \mathcal{L}^1(\mathbb{Q})$. This fact follows from

$$\mathbb{E}^{\mathbb{Q}}(|Y|) = \mathbb{E}^{\mathbb{P}}(|Y| \frac{d\mathbb{Q}}{d\mathbb{P}}) \leq || \frac{d\mathbb{Q}}{d\mathbb{P}} ||_{\infty}\ \mathbb{E}^{\mathbb{P}}(|Y|) < \infty.$$

Finally, set

$$\mathscr{C} = \{\mathbb{E}^{\mathbb{Q}}(Y) \mid \mathbb{Q} \in \mathscr{O}\} \subset \mathbb{R}^d.$$

Then $\mathscr{C}$ is a convex set as it is the image of a linear map of a convex set. We need to show that

$$0 = (0, ..., 0) \in \mathscr{C}.$$

Theorem 2.8 *(Separating hyperplane theorem). Suppose $\mathscr{C} \subset \mathbb{R}^d$ is a convex, non-empty set and $p_0 \notin \mathscr{C}$. Then there exists $\eta \in \mathbb{R}^d$ so that*

$$\eta \cdot (c - p_0) \geq 0 \quad \forall c \in \mathscr{C}, \quad \textit{and} \quad \exists c_1 \in \mathscr{C} \ni \eta \cdot (c_1 - p_0) > 0.$$

Moreover if $\inf |c - p_0| > 0$, *then one may choose* η *so that* $\inf_{c \in \mathscr{C}} \eta \cdot (c - p_0) > 0$.

Proof. See for instance Föllmer & Schied Appendix 1. ∎

Assume $0 \notin \mathscr{C}$. Then, there exists $\vartheta \in \mathbb{R}^d$ so that

$$\vartheta \cdot x \geq 0 \quad \forall x \in \mathscr{C} \quad \text{and} \quad \vartheta \cdot x_0 > 0,$$

for some $x_0 \in \mathscr{C}$. This means that

$$\mathbb{E}^{\mathbb{Q}}(\vartheta \cdot Y) \geq 0 \quad \forall \mathbb{Q} \in \mathscr{O} \quad \text{and} \quad \mathbb{E}^{\mathbb{Q}_0}(\vartheta \cdot Y) > 0$$

for one $Q_0 \in \mathscr{O}$. Since $Q_0 \approx P$ we get

$$\mathbb{P}(Y \cdot \vartheta > 0) > 0.$$

We claim that

$$\mathbb{E}^{\mathbb{Q}}(\vartheta \cdot Y) \geq 0 \quad \forall \mathbb{Q} \in \mathscr{O} \ \Rightarrow \ \vartheta \cdot Y \geq 0 \quad \mathbb{P} - a.s. \tag{2.2.2}$$

Notice that (2.2.2) would imply that ϑ is an arbitrage opportunity and thus contradicting our assumption. Hence $0 \in \mathscr{C}$ and the theorem is proved.

Set $A := \{\omega : \vartheta \cdot Y(\omega) < 0\}$ and

$$\varphi_n(\omega) := (1 - \frac{1}{n})\chi_A(\omega) + \frac{1}{n}\chi_{A^c}(\omega).$$

Further set

$$\mathbb{Q}^n(B) = \frac{\mathbb{E}^{\mathbb{P}}(\varphi_n \chi_B)}{\mathbb{E}^{\mathbb{P}}(\varphi_n)}, \quad \forall B \in \mathcal{F},$$

or equivalently

$$\frac{d\mathbb{Q}^n}{d\mathbb{P}} = \frac{\varphi_n}{\mathbb{E}^{\mathbb{P}}(\varphi_n)}.$$

Since $\varphi^n \geq 1/n$ for all $n \geq 2$, $\mathbb{Q}^n \in \mathscr{O}$. Therefore,

$$0 \leq \mathbb{E}^{\mathbb{Q}^n}(\vartheta \cdot Y) = \frac{1}{\mathbb{E}^{\mathbb{P}}(\varphi_n)} \mathbb{E}^{\mathbb{P}}(\varphi_n \vartheta \cdot Y).$$

We now use the dominated convergence theorem to obtain,

$$\mathbb{E}^{\mathbb{P}}(\vartheta \cdot Y \ \chi_A) = \lim_{n \uparrow \infty} \mathbb{E}^{\mathbb{P}}(\vartheta \cdot Y \varphi_n) \geq 0.$$

Hence $\vartheta \cdot Y \geq 0$, $\mathbb{P}$-a.s, proving (2.2.2).

Now consider the general case when Y is not necessarily in $\mathcal{L}^1(\mathbb{P})$. Define $\tilde{\mathbb{P}}$ by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}(\omega) = \frac{c}{1 + |S(\omega)|}, \quad c = \left[\mathbb{E}^{\mathbb{P}}\left(\frac{1}{1 + |S|}\right)\right]^{-1}.$$

Then $S \in \mathcal{L}^1(\tilde{\mathbb{P}})$ and $\tilde{\mathbb{P}} \approx \mathbb{P}$.

No-arbitrage under $\mathbb{P}$ implies no-arbitrage under $\tilde{\mathbb{P}}$. Hence there is $\mathbb{Q} \approx \tilde{\mathbb{P}}$ and S is a $\mathbb{Q}$ martingale. Moreover, $d\mathbb{Q}/d\tilde{\mathbb{P}}$ is bounded. Consequently,

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$$

is also bounded and $Q \approx P$. ∎

Lecture 5, October 6, 2011

2.3 Multiperiod Models

We want to obtain a similar result as in one-step case. This can be done by induction. However, in the previous proof $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and in induction this will be lost. So we first redo the one step with a general $\mathcal{F}_0$. See section 1.6 in Föllmer & Schied.

In this context, an arbitrage opportunity is a random $\mathcal{F}_0-$mbl vector $(\eta, \vartheta) \in \mathbb{R} \times \mathbb{R}^d$ so that

$$\eta + \vartheta \cdot S_0 \leq 0, \quad \eta + \vartheta \cdot S_1 \geq 0, \quad \mathbb{P} - a.s.,$$

and

$$\mathbb{P}(\eta + \vartheta \cdot S_1 > 0) > 0.$$

We set

$$Y := S_1 - S_0 \in \mathbb{R}^d,$$
$$K := \{\vartheta \cdot Y \mid \vartheta \text{ is } \mathcal{F}_0 - \text{mbl}\}$$

to be the set of all possible gains. Recall that in the simpler one-step proof we simply worked with $\mathscr{C} = \{E_Q Y\}$.) With this notation, there is no arbitrage if and only if $K \cap \mathcal{L}^0_+ = \{0\}$, where for any p,

$$\mathcal{L}^p_+ := \{X \mid X \text{ is } \mathcal{F}_1 - \text{mbl and } E|X|^p < \infty, X \geq 0 \ \mathbb{P} - a.s.\}$$

The necessary and sufficient condition for no-arbitrage is proved in the following theorem.

Theorem 2.9 (Thm 1.54 in F & S) *The following are equivalent,*

(*a*) $K \cap \mathcal{L}^0_+ = \{0\}$,

(*b*) $(K \oplus \{-\mathcal{L}^0_+\}) \cap \mathcal{L}^0_+ = \{0\}$,

(*c*) *There exists an equivalent martingale measure* $\mathbb{Q}$ *such that* $d\mathbb{Q}/d\mathbb{P}$ *is bounded,*

(*d*) *There exists an equivalent martingale measure* $\mathbb{Q}$.

Proof. (d) $\Rightarrow$ (a) done twice already. Also it is clear that (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (d).

To prove (a) $\Rightarrow$ (b), let $Z \in (K - \mathcal{L}^0_+) \cap \mathcal{L}^0_+$. Then there is $U \geq 0$ random variable in $\mathcal{L}^0_+$ and $\vartheta \in \mathcal{L}^0(\mathcal{F}_0)$ so that

$$Z = \vartheta \cdot Y - U \geq 0 \quad \mathbb{P} - a.s.$$

Hence

$$\vartheta \cdot Y \geq U \geq 0 \quad \mathbb{P} - a.s.$$

But (a) implies that $\vartheta \cdot Y = 0$ and $U = 0$.

So the only implication left to prove is (b) $\Rightarrow$ (c). This is done in 3 steps.

We first assume that

$$\mathbb{E}^{\mathbb{P}}(|S_0|), \ \mathbb{E}^{\mathbb{P}}(|S_1|) < \infty. \tag{2.3.3}$$

Set $\mathscr{C} := (K - \mathcal{L}^0_+) \cap \mathcal{L}^1$.

Lemma 2.10 (Lemma 1.58 in F & S) *Assume that $\mathscr{C}$ is closed in $\mathcal{L}^1$ and suppose that $\mathscr{C} \cap \mathcal{L}^1_+ = \{0\}$. Then for every nonzero $F \in \mathcal{L}^1_+$ there exists $Z^*_F \in \mathcal{Z}$ so that $E_P(FZ^*_F) > 0$, where*

$$\mathcal{Z} = \{Z \in L^\infty(\mathcal{F}_1, \mathbb{P}) \ : \ 0 \le Z \le 1, \ \mathbb{P}(Z > 0) > 0 \text{ and } \mathbb{E}^{\mathbb{P}}(ZW) \le 0 \ \forall W \in \mathscr{C}\}.$$

Proof. Let $\mathcal{B} = \{F\}$ where F is given point in $\mathcal{L}^1_+$. Since $F \neq 0$,

$$\mathcal{B} \cap \mathscr{C} = \emptyset.$$

Moreover, $\mathscr{C}$ is non-empty, convex and closed (by assumption). Thus, by Hahn-Banach theorem (see Thm A.56 in F & S, for instance) there is a linear, continuous functional ℓ on $\mathcal{L}^1(\mathcal{F}_1, \mathbb{P})$ so that

$$\sup_{W \in \mathscr{C}} \ell(W) < \ell(F).$$

Since the dual of $(\mathcal{L}^1)^* = \mathcal{L}^\infty$, there exists $\hat{Z} \in \mathcal{L}^\infty(\mathbb{P}, \mathcal{F}_1)$ so that

$$\ell(W) = \textstyle\int ZW d\mathbb{P} = \mathbb{E}^{\mathbb{P}}(ZW), \quad \forall W \in \mathcal{L}^1.$$

Set $Z^*_F = \hat{Z}/||\hat{Z}||_\infty$, then we have $||Z||_\infty \le 1$. Therefore,

$$\mathbb{E}^{\mathbb{P}}(WZ^*_F) < \mathbb{E}^{\mathbb{P}}(FZ^*_F) \quad \forall W \in \mathscr{C}. \tag{2.3.4}$$

We *claim* that above implies that $Z^*_F \in \mathcal{Z}$.

Proof of the claim: (Lemma 1.57 in F & S) For any $\lambda \ge 0$ and $W \in \mathscr{C}$, we have $\lambda W \in \mathscr{C}$. Then, if there is $W_0 \in \mathscr{C}$ so that $\mathbb{E}^{\mathbb{P}}(W_0 Z) > 0$, then

$$\sup_{\mathscr{C}} \mathbb{E}^{\mathbb{P}}(WZ^*_F) = +\infty$$

Hence

$$\sup_{W \in \mathscr{C}} \mathbb{E}^{\mathbb{P}}(WZ^*_F) \le 0.$$

Moreover, since $0 \in \mathscr{C}$, $\mathbb{E}^{\mathbb{P}}(FZ^*_F) > 0$.

Set $W := -\chi_{\{Z^*_F < 0\}}$. Then $W \in \mathscr{C}$ (this is the reason for working with $\mathscr{C} = (K - \mathcal{L}^0_+) \cap \mathcal{L}^1$ and not $K \cap \mathcal{L}^1$). Hence, by the construction of Z^*,

$$\mathbb{E}^{\mathbb{P}}(WZ^*_F) = - \ \mathbb{E}^{\mathbb{P}}(Z^*_F \chi_{\{Z^*_F < 0\}}) \le 0,$$

which implies that

$$\mathbb{P}(Z^*_F < 0) = 0.$$

We know that (2.3.4) excludes the possibility of $Z^*_F \equiv 0$. Hence

$$\mathbb{P}(Z^*_F > 0) > 0.$$

Thus $Z^*_F \in \mathcal{Z}$. ∎

For any $Z \in \mathcal{Z}$, set

$$\mathbb{P}_Z(A) := \frac{\mathbb{E}^{\mathbb{P}}(Z\chi_A)}{\mathbb{E}^{\mathbb{P}}(Z)}, \quad \forall A \in \mathcal{F}_1,$$

i.e.,

$$\frac{d\mathbb{P}_Z}{d\mathbb{P}} = \frac{Z}{\mathbb{E}^{\mathbb{P}}(Z)}.$$

Notice for any $\lambda \in \mathbb{R}$ and $\vartheta \in L^\infty(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$,

$$W := \lambda\vartheta \cdot (S_1 - S_0) = \lambda\vartheta \cdot Y \in \mathscr{C}$$

Hence,

$$\mathbb{E}^{\mathbb{P}}(Z_F^* W) = \lambda \mathbb{E}^{\mathbb{P}}(Z_F^* \vartheta \cdot Y) \leq 0 \quad \forall \lambda \in \mathbb{R}.$$

Since this holds for every $\lambda \in \mathbb{R}$ (not only $\lambda > 0$) we conclude that

$$\mathbb{E}^{\mathbb{P}}(Z(\vartheta \cdot Y)) = 0 \quad \forall \vartheta \in \mathcal{L}^\infty(\mathcal{F}_0, \mathbb{P}), \quad \Rightarrow \quad \mathbb{E}^{\mathbb{P}_Z}(\vartheta \cdot Y) = 0.$$

This implies that S is a $\mathbb{P}_Z$ martingale.

Moreover, $\mathbb{P}_Z \approx \mathbb{P}$ if $Z > 0$ almost surely. For this we need to show, there exist $Z \in \mathcal{Z}$ so that $\mathbb{P}(Z = 0) = 0$.

We will first show that there exists $Z^* \in \mathcal{Z}$ that is a maximizer for the following maximization problem, Set

$$c := \sup_{Z \in \mathcal{Z}} \mathbb{P}(\{Z > 0\}).$$

Choose $Z^n \in \mathcal{Z}$ so that $\mathbb{P}(Z^n > 0) \to c$ and define

$$Z^* := \sum_{n=1}^\infty 2^{-n} Z^n \;\; \Rightarrow \;\; Z^* \in \mathcal{Z} \quad \text{exercise.}$$

Then

$$\{Z^* > 0\} = \bigcup_n \{Z^n > 0\} \quad (\text{since all } Z^n \geq 0)$$

and $\mathbb{P}(Z^* > 0) = c$.

Now suppose that $\mathbb{P}(Z^* = 0) = 1 - c > 0$. Let

$$F = \chi_{\{Z^* = 0\}} \in \mathcal{L}^1_+ \text{ and } F \neq 0.$$

Then, as in Lemma 2.10, there exists Z_F^* so that

$$\mathbb{E}^{\mathbb{P}}(Z_F^* \cdot F) > 0 \Rightarrow \quad \mathbb{P}(\{Z_F^* > 0\} \cap \{Z^* = 0\}) > 0.$$

Then,

$$\mathbb{P}(\{\frac{1}{2}(Z^* + Z_F^*) > 0\}) = \mathbb{P}(Z^* > 0) + P(\{_F^* > 0\} \cap \{Z^* = 0\}) > c.$$

This contradicts with the fact that c the supremum. Hence c must be equal to one and the martingale measure $\mathbb{Q} := \mathbb{P}_{Z^*}$ is equivalent to $\mathbb{P}$.

Thus (b) $\Rightarrow$ (c) is proved under the additional assumptions,

(a) $S_0, S_1 \in \mathcal{L}^1$,

(b) $\mathscr{C}$ is closed.

We remove the first assumption as in the previous section. Indeed, let $\tilde{\mathbb{P}}$ be defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = c(1 + |S_0| + |S_1|)^{-1}$$

and proceed as in the $\mathcal{F}_0 = \{\emptyset, \Omega\}$ case.

Remark 2.11 In fact in the above we have proved the following useful theorem.

Theorem 2.12 (Kreps-Yan) *Suppose $\mathscr{C}$ is a closed, convex cone in $\mathcal{L}^1$ satisfying*

$$\mathscr{C} \supset -\mathcal{L}^\infty_+ \text{ and } \mathscr{C} \cap \mathcal{L}^1_+ = \{0\}$$

Then there is $Z^* \in \mathcal{L}^\infty$ so that $Z^* > 0$, a.s, and $\mathbb{E}(WZ^*) \le 0$ for all $W \in \mathscr{C}$.

In the next lecture we prove the closedness of $\mathscr{C}$. However, this is a subtle property and we give the next example to illustrate this point.

Example 2.13 This example illustrates that to prove the closedness of $\mathscr{C} = (K - \mathcal{L}^0_+) \cap \mathcal{L}^1$, the assumption that $\mathscr{C} \cap \mathcal{L}^1_+ = \{0\}$ is needed.

Indeed, let $\mathbb{P}$ be the Lebesgue measure on $[0,1] = \Omega$, $\mathcal{F}_1$ be the Borel σ-algebra, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $d = 1$ and $Y(\omega) = S_1(\omega) - S_0 = \omega$. Note that

$$K := \{\vartheta\, Y(\omega) \mid \vartheta \in \mathbb{R}^d\} \subset \mathcal{L}^1_+$$

So $(K - \mathcal{L}^0_+) \cap \mathcal{L}^0_+ \supseteq K \supsetneq \{0\}$.

On the other hand

$$\mathscr{C} = (K - \mathcal{L}^0_+) \cap \mathcal{L}^1 \neq \mathcal{L}^1.$$

See F & S (or exercise) for the strict inclusion. However, we will show that $\overline{\mathscr{C}} = \mathcal{L}^1$. Indeed, let $F \in \mathcal{L}^1$ be arbitrary and set

$$F_n := (F^+ \wedge n)\chi_{[\frac{1}{n},1]} - F^-.$$

Then,

$$\mathbb{E}^{\mathbb{P}}(|F - F_n|) = \textstyle\int_0^{1/n} F^+(\omega)d\omega + \int_{1/n}^1 \chi_{\{F^+ \ge n\}} F^+ d\omega.$$

Hence, $F_n \to F$ in $\mathcal{L}^1$. Moreover,

$$F_n \le (F^+ \wedge n)\chi_{[\frac{1}{n},1]} \le n\chi_{[\frac{1}{n},1]} \le n^2\omega\chi_{[\frac{1}{n},1]} \le n^2 Y(\omega).$$

Hence,

$$F_n = n^2 Y(\omega) - (n^2 Y(\omega) - F_n) \in K - \mathcal{L}^0_+$$

Summarizing, for any arbitrary $F \in \mathcal{L}^1$, we constructed a sequence $F_n \in \mathscr{C}$ and F_n converges to F. Therefore, the closure of $\mathscr{C}$ is the whole $\mathcal{L}^1$. Since $\mathscr{C} \neq \mathcal{L}^1$, we conclude that it is not closed.

So care is needed. ∎

Lecture 6, October 10, 2011

Claim: Assume that $\mathscr{C} \cap \mathcal{L}^1_+ = \{0\}$, then $\mathscr{C}$ is closed.
Proof of the claim:
Suppose $W_n \in \mathscr{C}$ converges to W in $\mathcal{L}^1$ and almost surely. Then,

$$W_n = \xi_n \cdot Y - U_n$$

for some $\xi_n \in \mathcal{L}^0(\mathcal{F}_0, \mathbb{P})$, $U_n \geq 0$ and $U_n \in \mathcal{L}^0(\mathcal{F}_1, \mathbb{P})$.
Step 1. We show that

$$\mathbb{P}(\liminf_{n\to\infty} |\xi_n| = +\infty) = 0.$$

Indeed, set

$$A := \{\liminf |\xi_n| = +\infty\}, \qquad \hat{\xi}_n = \frac{\xi_n}{|\xi_n|},$$

and when $\xi_n = 0$ then we arbitrarily set $\hat{\xi}_n := 1$. Then, there exists an $\mathcal{F}_0$ measurable subsequence σ_m so that

$$\hat{\xi}_{\sigma_m(\omega)} \to \hat{\xi}(\omega) \quad \mathbb{P}-a.s.,\ \omega \in \Omega$$

(See Lemma 1.63 in F & S). We use this convergence to conclude

$$0 \leq \chi_A \frac{U_{\sigma_m}}{|\xi_{\sigma_m}|} = \chi_A(\hat{\xi}_{\sigma_m} \cdot Y - \frac{W_{\sigma_m}}{|\xi_{\sigma_m}|}) \to \chi_A\ \hat{\xi} \cdot Y \quad \mathbb{P}-a.s.$$

Since $\chi_A\ \xi \cdot Y \in K$ and since, by assumption, $K \cap \mathcal{L}^0_+ = \{0\}$, we conclude that $\chi_A\ \hat{\xi} \cdot Y = 0$. Hence, $\chi_A = 0$ unless $\hat{\xi} \cdot Y = 0$. In general, we need to decompose the space in an obvious manner (See Lemma 1.65 in F & S) to conclude that $\chi_A = 0$. Hence,

$$\liminf_{n\to\infty} |\xi_n| < \infty \qquad \mathbb{P}-a.s.,$$

and there is σ_m (possibly different than the above) so that $\xi_{\sigma_m} \to \xi$ almost surely. Then,

$$\begin{aligned} W_{\sigma_m} &= \xi_{\sigma_m} \cdot Y - U_{\sigma_m} \to W \\ &\Rightarrow U_{\sigma_m} = -W_{\sigma_m} + \xi_{\sigma_m} \cdot Y \to\ -W + \xi \cdot Y =: U \\ &\Rightarrow W = \xi \cdot Y - U \in K - \mathcal{L}^0_+ \\ &\Rightarrow \mathscr{C} = (K - \mathcal{L}^0_+) \cap \mathcal{L}^1 \text{ is closed in } \mathcal{L}^1 \end{aligned}$$

■

We refer to F & S (end of chapter 1), for several useful comments on closure of those types of sets.

We now continue by proving the general N-step no-arbitrage theorem. The general structure is as follows,

We now assume that $S_0, ...S_T$ are given as $\mathbb{R}^d$ valued random variables.. To simplify we assume that

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \text{ and } \mathcal{F}_T = \mathcal{F}.$$

Definition 2.14 : A self-financing strategy (V_0, ϑ) is an *arbitrage opportunity* if.

$$V_0 \leq 0,\ V_T \geq 0\ \ \mathbb{P} - a.s.\ \ \mathbb{P}(V_T > 0) > 0$$

We start with an easy observation whose proof is left as an exercise (or see Proposition 5.11 in F & S).

Lemma 2.15 *An arbitrage opportunity exists if and only if there exists $t \in \{1, ..., T\}$ and $\eta \in \mathcal{L}^0(\mathcal{F}_{t-1})$ so that*

$$\eta \cdot (S_t - S_{t-1}) \geq 0 \quad \mathbb{P} - a.s.$$

and

$$\mathbb{P}(\eta \cdot (S_t - S_{t-1}) > 0) > 0.$$

We are now ready to prove the theorem.

Theorem 2.16 (Thm 5.17 in F& S) *The model admits no arbitrage opportunities if and only if there exists an EMM, $\mathbb{Q}$ with a bounded $d\mathbb{Q}/d\mathbb{P}$.*

Proof. Sufficiency has been proved. For necessity, set

$$K_t := \{\eta \cdot (S_t - S_{t-1}) \mid \eta \in \mathcal{L}^\infty(\mathcal{F}_{t-1})\}$$

Above lemma implies that $K_t \cap \mathcal{L}^0_+(\mathcal{F}_t) = \{0\}$, for all $t = 1, ..., T$. Then, we first use Theorem 2.9 at $t = T$ to obtain $\mathbb{Q}_T \approx \mathbb{P}$ so that

$$\frac{d\mathbb{Q}_T}{d\mathbb{P}} \in \mathcal{L}^\infty(\mathcal{F}_T) \quad \text{and} \quad \mathbb{E}^{\mathbb{Q}_T}(S_T - S_{T-1}|\mathcal{F}_{T-1}) = 0.$$

Towards a proof by induction, assume that for $t < T$ a probability measure $\mathbb{Q}_{t+1} \approx \mathbb{P}$ with the following property is constructed,

$$\mathbb{E}^{\mathbb{Q}_{t+1}}(S_k - S_{k-1}|\mathcal{F}_{k-1}) = 0, \quad t + 1 \leq k \leq T.$$

Now, since $\mathbb{Q}_{t+1} \approx \mathbb{P}$, we have $K_{t+1} \cap \mathcal{L}^0_{t+1}(\mathcal{F}_{t+1}, \mathbb{Q}_{t+1}) = \{0\}$ as well. We now apply Theorem 2.9 again at time t with $\mathbb{Q}_{t+1}$ instead of $\mathbb{P}$, to obtain an $\mathcal{F}_t$ measurable density Z_t so that

$$Z_t =: \frac{d\mathbb{Q}_t}{d\mathbb{Q}_{t+1}} \in \mathcal{L}^\infty(\mathcal{F}_t) \quad \text{and} \quad \mathbb{E}^{\mathbb{Q}_t}(S_t - S_{t-1}|\mathcal{F}_{t-1}) = 0.$$

Then, clearly

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}} = \frac{d\mathbb{Q}_t}{d\mathbb{Q}_{t+1}} \frac{d\mathbb{Q}_{t+1}}{d\mathbb{P}} \in \mathcal{L}^\infty.$$

For $k \geq t + 1$, using the $\mathcal{F}_t$ measurability of Z_t, we directly calculate that,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_t}(S_k - S_{k-1}|\mathcal{F}_{k-1}) &= \frac{\mathbb{E}^{\mathbb{Q}_{t+1}}((S_k - S_{k-1})Z_t|\mathcal{F}_{k-1})}{\mathbb{E}^{\mathbb{Q}_{t+1}}(Z_t|\mathcal{F}_{k-1})} \\ &= \mathbb{E}^{\mathbb{Q}_{t+1}}(S_k - S_{k-1}|\mathcal{F}_{k-1}) = 0. \end{aligned}$$

∎

The above theorem called “Dalang-Morton-Willinger” Theorem. “Equivalent martingale measures and no-arbitrage in stochastic securities market models, Stochastics and Stochastics Reports, 29. 185-201,1990.”

Continuous time:
No arbitrage-condition (NA) was stated as

$$
\begin{aligned}
G_T(\Theta_{adm}) \cap \mathcal{L}^0_+ = \{0\},& \\
&\Leftrightarrow \ (G_T(\Theta_{adm}) - \mathcal{L}^0_+) \cap \mathcal{L}^\infty \cap \mathcal{L}^0_+ = \{0\}, \\
&\Leftrightarrow \ \mathscr{C} \cap \mathcal{L}^0_+ = \{0\} \text{ where } \mathscr{C} := (G_T(\Theta_{adm}) - \mathcal{L}^0_+) \cap \mathcal{L}^\infty.
\end{aligned}
$$

However, for a general necessary and sufficient conditions, one needs to generalize both the no-arbitrage condition and also relax the martingale property. We start with a generalization of the definition of no-arbitrage.

Definition 2.17 A semimartingale S has the *no free lunch with vanishing risk* (NFLVR) property if

$$\overline{\mathscr{C}}^{\mathcal{L}^\infty} \cap \mathcal{L}^0_+ = \{0\} \quad \text{closure in} \mathcal{L}^\infty(\mathbb{P}, \mathcal{F}_T)).$$

We also recall that convergence in $\mathcal{L}^0$ is convergence in probability, i.e., $\xi_n \to \xi$ in $\mathcal{L}^0(\mathbb{P})$ means that for every $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|\xi_n - \xi| \geq \epsilon) = 0.$$

Proposition 2.18 *For S a semimartingale, the following are equivalent:*

1. *S has the no free lunch with vanishing risk property;*
2. *For any $g_n = G_T(\vartheta_n) \in G_T(\Theta_{adm})$ with*
$$G^-_T(\vartheta_n) \to 0 \text{ in } \mathcal{L}^\infty \quad \Rightarrow \quad G_T(\vartheta_n) \to 0 \text{ in } \mathcal{L}^0;$$
3. *S satisfies (NA) and the set*
$$\mathcal{G}^1 := \{G_T(\vartheta) \ : \ \vartheta \in \Theta_{adm}, \ G_\cdot(\vartheta) \geq -1\}$$
is bounded in $\mathcal{L}^0$, i.e.,
$$\lim_{n\to infty} \sup_{g\in\mathcal{G}^1} \mathbb{P}(|g| \geq n) = 0;$$
4. *S satisfies (NA) and for any $\varepsilon_n > 0$ converging to zero and ϑ_n satisfying $G_\cdot(\vartheta^n) \geq -\varepsilon_n$, we have $G_T(\vartheta^n) \to 0$ in $\mathcal{L}^0$.*

Next we recall a definition from stochastic processes.

Definition 2.19 A $\mathbb{R}^d$-valued process X is called a $\sigma-$martingale under $(\mathbb{P}, \mathcal{F}_t)$, if $X = \int \Psi dM$ for an $\mathbb{R}^d$-valued local martingale M and an $\mathbb{R}$-valued predictable, $M-$integrable process Ψ with $\Psi > 0$.

In general,

$$\text{Martingales} \subsetneq \text{ Local Mart. } \subsetneq \sigma\text{-Mart.}$$

Remark 2.20 Any $\sigma-$martingale X that is uniformly bounded from below by a deterministic constant is a local martingale. (A result of Ansel-Stricker).

Theorem 2.21 (Fundamental Theorem of Asset Pricing, Dalbean & Schachermayer 94-98 Math. Ann.) *For a semi-martingale $S = (S_t)_{0 \le t \le T}$, TFAE:*

1. *S satisfies (NFLVR);*

2. *S admits an equivalent separating measure $\mathbb{Q}$, i.e., $\mathbb{Q} \approx \mathbb{P}$ and*

$$\mathbb{E}^{\mathbb{Q}}[G_T(\vartheta)] \le 0, \quad \forall \vartheta \in \Theta_{adm};$$

3. *S admits an equivalent σ-martingale measure $\mathbb{Q}$, i.e, $\mathbb{Q} \approx \mathbb{P}$, and S is a $\mathbb{Q}$ σ-martingale.*

Proof. 3) $\Rightarrow$ 1) as before.

1) $\Rightarrow$ 2) : (NFLVR) $\Rightarrow \mathscr{C} = (G_T(\Theta_{adm}) - \mathcal{L}^0_+) \cap \mathcal{L}^\infty$ is weak-star closed. Then, by Kreps-Yan we construct the separating measure. But the weak-star closure proof is demanding.

2) $\Rightarrow$ 1): If S is locally bounded, easy. In general, the separating measure need not to be σ-martingale. But a density argument is used to complete the proof.

$\blacksquare$

Chapter 3

Black & Scholes Theory

Lecture 7, October 13, 2011
by Mario Sikic

In this section, we consider the classical Black & Scholes model. In this model, the stock price process is taken to be a geometric Brownian motion.

The basic references are:

1. F. Black, M. Scholes. *The pricing of options and corporate liabilities.* Journal of Political Economy, 81:637–654, 1976.

2. R. Merton. *An intertemporal capital asset pricing model.* Econometrica, 41:867–888, 1973.

Merton and Scholes received Nobel prize for their work in 1997. Black died two years before that in 1995.

3.1 Basic model

To define the model, set the finite horizon $T < \infty$ and a probabilty space $(\Omega, \mathcal{F}, P)$ on which there is a Brownian motion $(W_t)_{t\in[0,T]}$. We use the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by our Brownian motion W, augmented to satisfy the usual conditions under P.

Black–Scholes market model includes two assets: a riskless asset $(\tilde{B}_t)_{t\in[0,T]}$ and a risky asset $(\tilde{S}_t)_{t\in[0,T]}$. The model is described with the following equations

$$\begin{aligned} d\tilde{B}_t &= \tilde{B}_t r dt \\ d\tilde{S}_t &= \tilde{S}_t(\mu dt + \sigma dW_t) \end{aligned} \tag{3.1.1}$$

under the historical probability measure P. We denote the initial values to be $\tilde{S}_0$ and $\tilde{B}_0$. We call r the risk-free interest rate, μ is the drift, and $\sigma > 0$ the volatility. We choose these parameters to be constant. Solve this equation to obtain an explicit formulas for the processes

$$\begin{aligned} \tilde{B}_t &= \tilde{B}_0\, e^{rt} \\ \tilde{S}_t &= \tilde{S}_0\, e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}. \end{aligned} \tag{3.1.2}$$

Claim 1 The Black–Scholes market model is arbitrage free.

Proof. No-arbitrage condition is defined in discounted terms. The discounted processes are given with

$$\begin{aligned} S_t &= \tilde{S}_t/\tilde{B}_t = S_0\, e^{\sigma W_t + (\mu - r - \frac{\sigma^2}{2})t} \\ B_t &= 1. \end{aligned}$$

Recall that the process $M_t = \exp(\sigma W_t - \frac{\sigma^2}{2}t)$ is a martingale for any constant $\sigma > 0$. The idea is now to transform our discounted stock price process to get it into the form above. So, rearange the formula for the stock price as follows

$$\tilde{S}_t = \tilde{S}_0 \exp\Big(\sigma(W_t + \frac{\mu - r}{\sigma}t) - \frac{\sigma^2}{2}t\Big).$$

By Girsanov theorem, there exists an equivalent measure $Q \approx P$ under which the process $\hat{W}_t = W_t + \frac{\mu-r}{\sigma}t$ is a Brownian motion. Measure Q is defined via the density process

$$\mathbb{E}\Big[\frac{dQ}{dP}\Big|\mathcal{F}_t\Big] = Z_t = \mathcal{E}\Big(- \frac{\mu - r}{\sigma}W\Big)_t = \exp\Big(- \frac{\mu - r}{\sigma}W_t - \frac{(\mu - r)^2}{2\sigma^2}t\Big).$$

Or, equivalently, the Radon–Nikodym derivative of the measure change is given by

$$\frac{dQ}{dP} = \exp\Big(- \frac{\mu - r}{\sigma}W_T - \frac{(\mu - r)^2}{2\sigma^2}T\Big).$$

So, under the measure Q the process $\hat{W}$ is a Brownian motion and $S_0\, e^{\sigma \hat{W}_t - \frac{\sigma^2}{2}t}$ is a martingale. So, Q is an equivalent martingale measure for the discounted stock price process. ∎

Note 2 We call the quantity $\lambda = \frac{\mu-r}{\sigma}$ the **market price of risk**.

Itô representation theorem:

Every random variable $F \in L^1(\mathcal{F}_T^W, Q)$ admits a unique representation:

$$F = \mathbb{E}^Q[F] + \int_0^T H_s dW_s \qquad Q - a.s. \tag{3.1.3}$$

with a process $H \in L^2_{\text{loc}}(W)$, such that $(H \cdot W)_t$ is a martingale on $[0, T]$. (So, H is a predictable process, for which there is a sequence of stopping times $\tau_n \nearrow T$ such that for each n we have $\mathbb{E}[\int_0^{\tau_n} H_s^2 dt] < \infty$).

Consequently, every local P martingale N (with respect to Brownian filtration) is of the form

$$N_t = N_0 + \int_0^t H_s dW_s, \tag{3.1.4}$$

for some $H \in L^2_{\text{loc}}(W)$.

Claim 3 The equivalent martingale measure Q, defined as above, with the density

$$Z_t = \mathcal{E}\Big(-\frac{\mu - r}{\sigma}W\Big)_t$$

is unique.

Proof. Let Q' be a measure equivalent to P, and write the density process $Z'_t = \mathcal{E}(L')_t$. The process L' is a unique P-local martingale starting at zero and defined with $L'_t = \int_0^t \frac{1}{Z'_s} dZ'_s$. This is well defined, since Z'_t is strictly greater than 0 P-a.s. for all $t \in [0, T]$ simultaneously. Note that since Z' and L' are local martingales under the Brownian filtration, they have continuous versions. Use Itô representation theorem under P which gives a process H_s such that

$$L_t = \int_0^t H_s dW_s.$$

Now, assume that the discounted stock price process (S_t) is a Q' martingale. By Bayes rule, the process $(Z_t S_t)$ is a local P martingale. Hence, we calculate (under the measure P)

$$\begin{aligned} d(Z'_t S_t) &= Z'_t dS_t + S_t dZ'_t + d\langle Z', S_t\rangle_t \\ &= Z'_t S_t(\mu - r)dt + Z'_t S_t \sigma dW_t + S_t H_t dW_t + Z'_t H_t S_t \sigma d\langle W\rangle_t \\ &= Z'_t S_t(\sigma + H_t)dW_t + Z'_t S_t \sigma (H_t + \lambda)dt. \end{aligned}$$

Now, the left hand side is a local martingale, so also

$$A_t = \int_0^t Z'_t S_t \sigma (H_t + \lambda) dt$$

is a local martingale of finite variation, starting at zero. We conclude that $A_t = 0$. Since $Z'_t S_t \sigma > 0$, we conclude that $H_t = -\lambda$. ∎

3.2 Market completeness

Definition 4 Let $(\tilde{S}_t)_{t \in [0,T]}$ be a market model. We say that the market S is complete if every (reasonable) contingent claim $X \in L^0(\mathcal{F}_T)$ is replicable, i.e. there exist an initial wealth V_0 and a strategy ϑ such that the final wealth is

$$V_T(\varphi) = V_0 + \int_0^T \vartheta_t dS_t = X.$$

Claim 5 The Black–Scholes market model is complete, in the sense that every contingent claim X, such that $X/\tilde{B}_t$ is bounded from below and in $L^1(Q, \mathcal{F}_T)$ is replicable.

Proof. Let ϑ be a strategy (position in stock). We can write the dynamics of the discounted wealth process

$$
\begin{aligned}
d\Big(\frac{V_t}{\tilde{B}_t}\Big) &= \frac{1}{\tilde{B}_t} dV_t - \frac{V_t}{\tilde{B}_t^2} d\tilde{B}_t \\
&= \frac{1}{\tilde{B}_t}[\vartheta_t d\tilde{S}_t + \frac{V_t - \vartheta_t \tilde{S}_t}{\tilde{B}_t} d\tilde{B}_t] - \frac{V_t}{\tilde{B}_t^2} d\tilde{B}_t \\
&= \frac{1}{\tilde{B}_t}[\vartheta_t \tilde{S}_t \sigma dW_t + \vartheta_t \tilde{S}_t (\mu - r) dt] \\
&= \frac{\vartheta_t \tilde{S}_t \sigma}{\tilde{B}_t} d(W_t + \lambda t) \\
&= \frac{\vartheta_t \tilde{S}_t \sigma}{\tilde{B}_t} d\hat{W}_t,
\end{aligned}
$$

where we have used that the position in bond is $\eta_t = (V_t - \vartheta_t \tilde{S}_t)/\tilde{B}_t$ by the wealth equation. We got that the discounted wealth process is a Q local martingale.

Let the $\mathcal{F}_T$ measurable random variable X be as defined in the statement of the claim. Under Q, Itô theorem gives us a representation of the random variable $X/\tilde{B}_T$ of the following form

$$
\frac{X}{\tilde{B}_T} = \mathbb{E}^Q[X/\tilde{B}_T] + \int_0^T H_s d\hat{W}_s,
$$

such that the process $\int_0^T H_s d\hat{W}_s$ is a martingale. So, if we define a strategy as follows,

$$
\vartheta_t = \frac{H_t \tilde{B}_t}{\sigma \tilde{S}_t},
$$

we get that the value process exactly replicates the claim $X/\tilde{B}_t$ in the discounted market model.

Let's show that this strategy also replicates the claim X in the undiscounted market model. Write the dynamics of the undiscounted value process

$$
\begin{aligned}
dV_t &= \vartheta_t d\tilde{S}_t + r(V_t - \vartheta_t \tilde{S}_t) dt \\
&= r V_t dt + \vartheta_t \tilde{S}_t \sigma dW_t + \vartheta_t \tilde{S}_t (\mu - r) dt \\
&= r V_t dt + \vartheta_t \tilde{S}_t \sigma d(W_t + \lambda t) \qquad (3.2.5)
\end{aligned}
$$

And solve it to get

$$
\begin{aligned}
V_t &= e^{rt}\Big[V_0 + \int_0^t e^{-rt} \vartheta_t \tilde{S}_t \sigma d\hat{W}_s\Big] \\
&= \tilde{B}_t \Big[\frac{V_0}{\tilde{B}_0} + \int_0^t \vartheta_t \frac{\tilde{S}_t \sigma}{\tilde{B}_t} d\hat{W}_s\Big].
\end{aligned}
$$

So, using $V_0 = \tilde{B}_0 \, \mathbb{E}^Q[X/\tilde{B}_T]$ and strategy (ϑ) as defined above replicates the claim X.

What we still need to show is that the strategy ϑ defined above is admissible, i.e. that the wealth process is uniformly bounded below. Let $a \in \mathbb{R}$ be such that $X/\tilde{B}_T > a$ P-a.s. The discounted value process is a continuous martingale, replicates the claim $X/\tilde{B}_T$, and starts at $\mathbb{E}^Q[X/\tilde{B}_T] > a$. Hence, the entire discounted value process has to be greater than a. Since the process $\tilde{B}$ is uniformly bounded, the conclusion follows. ∎

3.3 Pricing and hedging of European claims

An European contingent claim X is a claim of the form $X = g(\tilde{S}_T)$ for some measurable function g.

In the proof of the claim above, we showed that the discounted value process is a Q martingale. Write that as follows

$$V_t = \mathbb{E}^Q\Big[\frac{V_T\tilde{B}_t}{\tilde{B}_t}\Big|\mathcal{F}_t\Big] = \mathbb{E}^Q[e^{-r(T-t)}V_T|\mathcal{F}_t]$$

Note that, since we have the process $\tilde{S}$ explicitely given, we can calculate the above conditional expectation

$$\begin{aligned} V_t &= \mathbb{E}^Q[e^{-r(T-t)}g(\tilde{S}_t)|\mathcal{F}_t] \\ &= \mathbb{E}^Q[e^{-r(T-t)}g(\tilde{S}_t\, e^{\sigma(W_T-W_t)+(\mu-\frac{\sigma^2}{2})(T-t)})|\mathcal{F}_t]. \end{aligned}$$

Now, take into account that W is an $\mathbb{F}$ Brownian motion, i.e. that $(W_T - W_t) \perp \mathcal{F}_t$. Calculate

$$V_t = \mathbb{E}^Q\Big[e^{-r(T-t)}g(se^{\sigma(W_T-W_t)+(\mu-\frac{\sigma^2}{2})(T-t)})\Big]\Big|_{s=\tilde{S}_t(\omega)} = v(t, \tilde{S}_t(\omega)),$$

for a measurable function v. In the above expectation, we now pass to the Brownian motion $\hat{W}_t$ under Q. Writting the function v out

$$\begin{aligned} v(t,s) &= \mathbb{E}^Q\Big[e^{-r(T-t)}g(se^{\sigma(\hat{W}_T-\hat{W}_t)+(r-\frac{\sigma^2}{2})(T-t)})\Big] \\ &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{\mathbb{R}} g\Big(se^{\sigma\sqrt{T-t}y+(r-\frac{\sigma^2}{2})(T-t)}\Big)e^{-\frac{y^2}{2}}\,dy. \end{aligned}$$

This last integral can be shown to be smooth on $(0,T)\times\mathbb{R}_+$ for a nice enough contingent claim.

Assume now that v is a sufficiently smooth function. It gives the value as a function of the stock price and time, and we know its dynamics. Use now the Itô formula

$$\begin{aligned} dv(t,\tilde{S}_t) &= v_t(t,\tilde{S}_t)dt + v_s(t,\tilde{S}_t)d\tilde{S}_t + \frac{1}{2}v_{ss}(t,\tilde{S}_t)d\langle\tilde{S}\rangle_t \\ &= v_t(t,\tilde{S}_t)dt + v_s(t,\tilde{S}_t)\tilde{S}_t\sigma dW_t + v_s(t,\tilde{S}_t)\tilde{S}_t\mu dt + \frac{\sigma^2 s^2}{2}v_{ss}(t,\tilde{S}_t)dt \\ &= \Big[v_t(t,\tilde{S}_t) + v_s(t,\tilde{S}_t)\tilde{S}_t\mu + \frac{\sigma^2 s^2}{2}v_{ss}(t,\tilde{S}_t)\Big]dt + v_s(t,\tilde{S}_t)\tilde{S}_t\sigma dW_t \end{aligned}$$

Compare this with the dynamics of the wealth process (3.2.5)

$$dV_t = [rV_t + \vartheta_t\tilde{S}_t(\mu - r)]dt + \vartheta_t\tilde{S}_t\sigma dW_t$$

to get from the dW term

$$\vartheta_t = v_s(t,\tilde{S}_t) \tag{3.3.6}$$

and from the dt term

$$v_t + v_s sr + \frac{\sigma^2 s^2}{2}v_{ss} = rv \tag{3.3.7}$$

given that $v(T,\cdot) = g(\cdot)$. This last equation called the Black–Scholes PDE.

3.4 The Feynman–Kac approach

We can, alternatively, arive at the same equation for the process v using the Feynman–Kac formula.

Feynman–Kac formula:

If $v(t,s)$ is a sufficiently smooth solution to the PDE

$$v_t + \mu s v_s + \frac{1}{2}\sigma^2 s^2 v_{ss} - rv = 0 \qquad \text{on } (0,T)\times\mathbb{R}, \tag{3.4.8}$$

$$v(T,\cdot) = g(\cdot) \qquad \text{on } \mathbb{R}, \tag{3.4.9}$$

then the function v is given by

$$v(t,s) = \mathbb{E}\Big[e^{-r(T-t)} g(\tilde{S}_t^{t,s})dt\Big], \tag{3.4.10}$$

where

$$\begin{aligned} d\tilde{B}_t &= \tilde{B}_t r dt \\ d\tilde{S}_t &= \tilde{S}_t(\mu dt + \sigma dW_t). \end{aligned}$$

We use the Feynman–Kac formula under the equivalent martingale measure.

Proposition 6 Let v be the sufficiently smooth solution to the Feynman–Kac PDE. Then the strategy, given with

$$\vartheta_t = v_s(t, \tilde{S}_t)$$

hedges the contingent claim $g(\tilde{S}_t)$, and is called the **delta hedge**.

Note 7 The greeks are:

- delta: v_s
- gamma: v_{ss}
- theta: v_t
- rho: v_r
- vega: v_σ

3.5 Examples

European call option. An european call option is a financial instrument with payoff at maturity given with $g(x) = (x-K)^+$. The number K is called strike, and T is maturity. So, in this case we have

$$v(t,s) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{\mathbb{R}} \Big(se^{\sigma y\sqrt{T-t} + (r-\frac{\sigma^2}{2})(T-t)} - K\Big)^+ e^{-\frac{y^2}{2}} dy$$

Proceeding with direct calculation, we get the celebrated Black–Scholes formula:

$$v(t,s) = s\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{3.5.11}$$

where

$$d_{1,2} = \frac{\log\left(\frac{s}{Ke^{-r(T-t)}}\right) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \tag{3.5.12}$$

and

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{x^2}{2}}\,dx. \tag{3.5.13}$$

One can show that the function v defined above is smooth for $t \neq T$. The hedging strategy is the delta hedge $\vartheta_t = v_s(t, \tilde{S}_t)$, where

$$v_s(t,s) = \Phi(d_1).$$

Digital option. The digital option is a contingent claim with the payoff $X = \mathbf{1}_{\{\tilde{S}_t > K\}}$. We calculate the value process directly

$$\begin{aligned}
V_t &= \mathbb{E}^Q[e^{-r(T-t)}\mathbf{1}_{\{\tilde{S}_t>K\}}|\mathcal{F}_t] \\
&= e^{-r(T-t)}Q[\tilde{S}_t e^{\sigma(W_T-W_t)+(\mu-\frac{\sigma^2}{2})(T-t)} > K|\mathcal{F}_t] \\
&= e^{-r(T-t)}Q[\tilde{S}_t e^{\sigma(\hat{W}_T-\hat{W}_t)+(r-\frac{\sigma^2}{2})(T-t)} > K|\mathcal{F}_t] \\
&= e^{-r(T-t)}Q[s e^{\sigma(\hat{W}_T-\hat{W}_t)+(r-\frac{\sigma^2}{2})(T-t)} > K]\big|_{s=\tilde{S}_t(\omega)} \\
&= e^{-r(T-t)}Q[\sigma(\hat{W}_T-\hat{W}_t) > \log\frac{K}{s} - (r-\frac{\sigma^2}{2})(T-t)]\big|_{s=\tilde{S}_t(\omega)} \\
&= e^{-r(T-t)}Q[\xi < \frac{\log\frac{s}{K} + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}]\big|_{s=\tilde{S}_t(\omega)} \\
&= e^{-r(T-t)}\Phi\Big[\frac{\log\frac{\tilde{S}_t}{K} + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\Big],
\end{aligned}$$

where ξ is a standard normal random variable. The delta hedge is in this case given with

$$\vartheta_t = v_s(t,\tilde{S}_t) = e^{-r(T-t)}\phi\Big[\frac{\log\frac{\tilde{S}_t}{K} + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\Big]\frac{1}{\tilde{S}_t\sigma\sqrt{T-t}}, \tag{3.5.14}$$

where $\phi = \Phi'$ is the density of the standard normal random variable.

3.6 Binomial tree approximation

For the binomial model, we saw that the value process evolves as

$$v(k\Delta t, s) = e^{-r\Delta t}\frac{1}{2}[v((k+1)\Delta t, s(1+\sigma\sqrt{\Delta t})) + v((k+1)\Delta t, s(1-\sigma\sqrt{\Delta t}))]$$

Expand the right hand side of this equation into its Taylor series around the point (s, t)

$$\begin{aligned}
v(t,s) \approx (1 - r\Delta t)\frac{1}{2}\Big[& \\
& v(t,s) + v_t(t,s)\Delta t + v_s(t,s)s\sigma\sqrt{\Delta t} + \frac{1}{2}v_{ss}s^2\sigma^2\Delta t + \\
& v(t,s) + v_t(t,s)\Delta t - v_s(t,s)s\sigma\sqrt{\Delta t} + \frac{1}{2}v_{ss}s^2\sigma^2\Delta t\Big] \\
= (1 - r\Delta t)v(t,s) + (1 - r\Delta t)\Big[v_t(t,s) + \frac{1}{2}v_{ss}s^2\sigma^2\Big]\Delta t &
\end{aligned}$$

Chapter 4

Quantile hedging

Lectures 8 and 9, October 17 and 20, 2011
by Erdinc Akyildirim

4.1 Introduction

- The problem of pricing and hedging of contingent claims is well understood in the context of arbitrage-free models which are complete. In such models every contingent claim is attainable, i.e., it can be replicated by a self-financing trading strategy. The cost of replication defines the price of the claim, and it can be computed as the expectation of the claim under the unique equivalent martingale measure.

- In an incomplete market the equivalent martingale measure is no longer unique, and not every contingent claim is attainable. There is an interval of arbitrage-free prices, given by the expected values under the different equivalent martingale measures. It is still possible to stay on the safe side by using a "superhedging" strategy, cf. El Karoui and Quenez (1995) and Karatzas (1997). The cost of carrying out such a strategy is given by the supremum of the expected values over all equivalent martingale measures. But in some situations the cost of superhedging can be too high from a practical point of view.

- What if the investor is unwilling to put up the initial amount of capital required by a perfect hedging or superhedging strategy? What is the maximal probability of a successful hedge the investor can achieve with a given smaller amount? Equivalently one can ask how much initial capital an investor can save by accepting a certain shortfall probability, i.e., by being willing to take the risk of having to supply additional capital at maturity in, e.g., 1% of the cases.

4.2 The Complete Market Case

4.2.1 Formulation of the problem

- We assume that the discounted price process of the underlying is given as a semimartingale $X = (X_t)_{t\in[0,T]}$ on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with filtration $(\mathcal{F})_{t\in[0,T]}$

- Let $\mathcal{P}$ denote the set of all equivalent martingale measures. We assume absence of arbitrage in the sense that $\mathcal{P} \neq \emptyset$.

- A self-financing strategy is defined by an initial capital V_0 and by a predictable process ξ which serves as an integrand for the semi-martingale X. Such a strategy (V_0, ξ) will be called admissible if the resulting value process V defined by

$$V_t = V_0 + \int_0^t \xi_s \, \mathrm{d}X_s \quad \forall t \in [0, T], \quad P - a.s \tag{4.2.1}$$

  satisfies

$$V_t \geq 0 \quad \forall t \in [0, T], \quad P - a.s. \tag{4.2.2}$$

- In the complete case there is a unique equivalent martingale measure $P^* \approx P$

- Consider a contingent claim given by a $\mathcal{F}_T$-measurable, nonnegative random variable H such that $H \in L^1(P^*)$

- Completeness implies that there exists a perfect hedge, i.e., a predictable process ξ^H such that

$$E^*[H \mid \mathcal{F}_t] = \underbrace{E^*[H]}_{H_0} + \int_0^t \xi_s^H \, \mathrm{d}X_s \qquad \forall t \in [0, T], \quad P - a.s \tag{4.2.3}$$

  where E^* denotes expectation with respect to P^*

- Thus the claim can be replicated by the self-financing trading strategy (H_0, ξ^H). This assumes, of course, that we are ready to allocate the required initial capital $H_0 = E^*[H]$.

- But what if the investor is unwilling or unable to put up the initial capital H_0? What is the best hedge the investor can achieve with a given smaller amount $m < H_0$? As our optimality criterion we take the probability that the hedge is successful. Thus we are looking for an admissible strategy (V_0, ξ) such that

$$P[V_T \geq H] = P\left[V_0 + \int_0^T \xi_s \, \mathrm{d}X_s \geq H\right] = max \tag{4.2.4}$$

  under the constraint

$$V_0 \leq m. \tag{4.2.5}$$

4.3 Maximizing the probability of success

Let us call the set $A = \{V_T \geq H\}$ the "success set" corresponding to the admissible strategy (V_0, ξ).

Proposition 1 Let P^* denote the unique equivalent martingale measure in a complete market. Assume that $\hat{A} \in \mathcal{F}_T$ maximizes the probability $P(A)$ among all sets $A \in \mathcal{F}_T$ satisfying the constraint

$$E^*[H \cdot 1_A] \leq m. \tag{4.3.6}$$

Let ξ^* denote the replicating strategy for the knock-out (KO) option $H^* = H \cdot 1_{\hat{A}}$. Then (m, ξ^*) solves the optimization problem defined by (4.2.4) and (4.2.5), and $\hat{A}$ coincides almost surely with the success set of (m, ξ^*).

Proof.

- Let V be the value process of any admissible strategy (V_0, ξ) such that $V_0 \leq m$. The success set for this strategy is denoted by $A = \{V_T \geq H\}$. Therefore, we know that on the set A
$$V_T = V_0 + \int_0^T \xi_s \, \mathrm{d}X_s \geq H, \tag{4.3.7}$$
which is equivalent to
$$V_T \cdot 1_A \geq H \cdot 1_A. \tag{4.3.8}$$
- Since (V_0, ξ) is an admissible strategy, we know that $V_T \geq 0$. If we take $\omega \notin A$ then $V_T \geq 0 = H \cdot 1_A$. But if we take $\omega \in A$ then by definition of A we have $V_T \geq H = H \cdot 1_A$. These together yield
$$V_T \geq H \cdot 1_A. \tag{4.3.9}$$
$$\Rightarrow E^*[V_T] \geq E^*[H \cdot 1_A]. \tag{4.3.10}$$
- X_t is a martingale under P^*, hence it is a local martingale under P^*. Then by a result of Ansel and Stricker Theorem, V_t is also a local martingale. We also know that every local martingale which is bounded from below is a super-martingale. Hence V_t is a super-martingale under P^* which by definition gives
$$V_0 \geq E^*[V_T] \tag{4.3.11}$$
- Using the assumption $V_0 \leq m$ with (4.3.10) and (4.3.11) gives
$$E^*[H \cdot 1_A] \leq E^*[V_T] \leq V_0 \leq m. \tag{4.3.12}$$
- The above inequality shows that the success set A corresponding to (V_0, ξ) satisfies the constraint (4.3.6) and thus by the maximality of $P(\hat{A})$ we get
$$P(A) \leq P(\hat{A}). \tag{4.3.13}$$
- Now we claim that any trading strategy with (V_0, ξ^*) with $E^*[H \cdot 1_{\hat{A}}] \leq V_0 \leq m$ is optimal. We first have to show that this strategy is admissible.
$$\begin{aligned} & V_0 \geq E^*[H \cdot 1_{\hat{A}}] \qquad (by \quad assumption) \\ \Leftrightarrow \; & V_0 + \int_0^t \xi_s^* \, \mathrm{d}X_s \geq E^*[H \cdot 1_{\hat{A}}] + \int_0^t \xi_s^* \, \mathrm{d}X_s \\ \Leftrightarrow \; & V_t^* \geq E^*[H \cdot 1_{\hat{A}} \mid \mathcal{F}_t] \\ \Leftrightarrow \; & V_t^* \geq 0 \text{ (Since H is a non-negative payoff, } H \cdot 1_{\hat{A}} \geq 0, \text{)} \end{aligned}$$
From here it follows that (V_0, ξ^*) is admissible.

- Let $A^* = \{V_T^* \geq H\} = \{V_0 + \int_0^t \xi_s^* \, \mathrm{d}X_s \geq H\}$ be the success set corresponding to (V_0, ξ^*) then we want to show that $A^* = \hat{A} \; P - a.s.$

- $P(A^*) \leq P(\hat{A})$ follows from (4.3.13).

- Let's show the other side of the equality i.e. $P(\hat{A}) \leq P(A^*)$.

$$\begin{aligned}
& V_0 \geq E^*[H \cdot 1_{\hat{A}}] \qquad (by \quad assumption) \\
\Leftrightarrow \; & V_0 + \int_0^T \xi_s^* \, \mathrm{d}X_s \geq E^*[H \cdot 1_{\hat{A}}] + \int_0^T \xi_s^* \, \mathrm{d}X_s \\
\Leftrightarrow \; & V_0 + \int_0^T \xi_s^* \, \mathrm{d}X_s \geq H \cdot 1_{\hat{A}} \\
\Leftrightarrow \; & V_T^* \geq H \cdot 1_{\hat{A}}.
\end{aligned}$$

  and the last inequality follows from

$$H \cdot 1_{\hat{A}} = E^*[H \cdot 1_{\hat{A}} \mid \mathcal{F}_T] = E^*[H \cdot 1_{\hat{A}}] + \int_0^T \xi_s^* \, \mathrm{d}X_s \tag{4.3.14}$$

- Let $\omega \in \hat{A}$, then $V_0 + \int_0^T \xi_s^* \, \mathrm{d}X_s \geq H$, which implies $\omega \in A^* = \{V_T^* \geq H\}$ and hence

$$\hat{A} \subseteq A^* \Rightarrow P(\hat{A}) \leq P(A^*)$$

- Therefore, any trading strategy (V_0, ξ^*) with $E^*[H \cdot 1_{\hat{A}}] \leq V_0 \leq m$ is optimal. In particular, (m, ξ^*) is optimal.

■

Our next goal is the construction of the optimal success set $\hat{A}$, whose existence was assumed in the Proposition (1). This problem is solved by using the Neyman-Pearson lemma.

Neyman-Pearson Lemma Take P and Q two probability measures on $(\Omega, \mathcal{F})$. If there exists A^0

$$A^0 = \left\{ \frac{dP}{dQ} > c \right\} \text{ for some } c \geq 0 \tag{4.3.15}$$

such that $Q(A) \leq Q(A^0) \qquad \forall A \in \mathcal{F_T}$, then $P(A) \leq P(A^0) \qquad \forall A \in \mathcal{F_T}$.

- Now define a new measure Q by

$$\frac{dQ}{dP^*} = \frac{H}{E^*[H]} = \frac{H}{H_0}. \tag{4.3.16}$$

- The constraint (4.3.6) can now be written as

$$\begin{aligned}
& E^*[H \cdot 1_A] \leq m \\
\Leftrightarrow\ & E^*\left[\frac{H \cdot 1_A}{H_0}\right] \leq \frac{m}{H_0} \\
\Leftrightarrow\ & E^*\left[\frac{dQ}{dP^*} \cdot 1_A\right] \leq \frac{m}{H_0} \\
\Leftrightarrow\ & E_Q[1_A] \leq \frac{m}{H_0} \\
\Leftrightarrow\ & Q(A) \leq \frac{m}{H_0} = \alpha
\end{aligned}$$

- Define the level

$$c^* = \inf\left\{c \geq 0 \mid Q\left[\frac{dP}{dQ} > c \cdot E^*[H]\right] \leq \alpha\right\}. \tag{4.3.17}$$

and the corresponding set

$$A^0 = \left\{\frac{dP}{dQ} > c^* \cdot E^*[H]\right\} = \left\{\frac{dP}{dP^*} > c^* \cdot H\right\} \tag{4.3.18}$$

Theorem 1 If the set $\hat{A}$ satisfies $Q(\hat{A}) = \alpha$ then the optimal strategy solving the optimization problem (4.2.4) and (4.2.5) is given by (m, ξ^*), where ξ^* is the replicating strategy of the knock-out option $H^* = H \cdot 1_{\hat{A}}$.

Proof. We know that P and Q are dominated by the equivalent martingale measure P^* and that $\hat{A}$ is of the form

$$\hat{A} = \left\{\frac{dP}{dP^*} > c^* \cdot H\right\}.$$

Using the Neyman-Pearson Lemma, $P(A) \leq P(\hat{A})$ holds for all sets $A \in \mathcal{F}_T$ such that $Q(A) \leq Q(\hat{A})$. Using that $Q(\hat{A}) = \alpha$ which means that the constraint $E[H \cdot 1_{\hat{A}}] = m$ is satisfied in Proposition (1), then the proof follows from the proposition. ■

4.4 Quantile hedging in the Black-Scholes model

- In the standard Black-Scholes model with constant volatility $\sigma > 0$, the underlying price process is given by a geometric Brownian Motion

$$dX_t = X_t(\mu dt + \sigma dW_t) \tag{4.4.19}$$

$$\Rightarrow \quad X_t = X_0 exp\left(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t\right) \tag{4.4.20}$$

where W is a Wiener process under P and μ is a constant. For simplicity we set the interest rate equal to zero.

- The unique equivalent martingale measure is then given by

$$\frac{dP^*}{dP} = \exp\left(-\frac{1}{2}(\frac{\mu}{\sigma})^2 T - \frac{\mu}{\sigma} W_T\right). \tag{4.4.21}$$

- The process W_t^* defined by
$$W_t^* = W_t + \frac{\mu}{\sigma}t \tag{4.4.22}$$
is a standard Brownian motion under P^* and
$$dX_t = \sigma X_t dW_t^* \tag{4.4.23}$$
$$\Rightarrow \quad X_t = X_0 exp\left(\sigma W_t^* - \frac{1}{2}\sigma^2 t\right) \tag{4.4.24}$$

- For quantile hedging, we want to (or can) pay an initial capital V_0 which is smaller than the Black-Scholes price H_0. The optimal strategy is to replicate the knock-out option $H \cdot 1_A$, where the set A is of the form $A = \{\frac{dP}{dP^*} > \text{const} \cdot H\}$.
Claim: $\frac{dP^*}{dP} = const \cdot X_T^{-\frac{\mu}{\sigma^2}}$
Proof:
$$\begin{aligned}
X_T^{-\frac{\mu}{\sigma^2}} &= x_0^{-\frac{\mu}{\sigma^2}} \cdot \exp(-\frac{\mu}{\sigma^2}(\sigma W_T + (\mu - \frac{\sigma^2}{2})\, T)) \\
&= x_0^{-\frac{\mu}{\sigma^2}} \cdot \exp(-\frac{\mu}{\sigma}W_T - \frac{\mu^2}{\sigma^2}T + \frac{\mu}{2}T) \\
&= x_0^{-\frac{\mu}{\sigma^2}} \cdot \exp(-\frac{\mu}{\sigma}W_T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T + \frac{\mu}{2}T) \\
&= x_0^{-\frac{\mu}{\sigma^2}} \cdot \exp(-\frac{1}{2}\frac{\mu^2}{\sigma^2}T + \frac{\mu}{2}T) \cdot \underbrace{\exp(-\frac{\mu}{\sigma}W_T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T)}_{\frac{dP^*}{dP}}.
\end{aligned}$$
Therefore, it follows that
$$\frac{dP^*}{dP} = \underbrace{\frac{1}{X_0^{-\frac{\mu}{\sigma^2}} \cdot \exp(-\frac{1}{2}\frac{\mu^2}{\sigma^2}T + \frac{\mu}{2}T)}}_{\frac{1}{\beta}} \cdot X_T^{-\frac{\mu}{\sigma^2}}, \tag{4.4.25}$$
where β is constant since μ, σ, T and X_0 are all constants. This leads to
$$\begin{aligned}
A &= \left\{\frac{dP}{dP^*} > \text{const} \cdot H\right\} \\
&= \{\beta \cdot X_T^{\frac{\mu}{\sigma^2}} > \text{const} \cdot H\} \\
&= \{X_T^{\frac{\mu}{\sigma^2}} > \lambda \cdot H\},
\end{aligned}$$
where λ is chosen such that $E^*[H \cdot 1_A] = V_0$.

4.5 European Call Option Example

- A European call $H = (X_T - K)^+$ can be hedged perfectly if we use the initial capital
$$H_0 = E^*[H] = x_0 \cdot \Phi(d_+) - K \cdot \Phi(d_-), \text{ where} \tag{4.5.26}$$
$$d_\pm = -\frac{1}{\sigma\sqrt{T}}\log(\frac{K}{x_0}) \pm \frac{1}{2}\sigma\sqrt{T}. \tag{4.5.27}$$

We distinguish two cases:

4.5.1 Case 1: $m \leq \sigma^2$

If $m \leq \sigma^2$ then $X_T^{\frac{m}{\sigma^2}}$ is a concave function.

As it is clear from the figure above the success set $A = \{X_T^{\frac{m}{\sigma^2}} > \lambda(X_T - K)^+\}$ corresponds to the set $A = \{X_T < c\}$, where c is the intersection point of two curves. By using the expression for X_T, we have

$$\begin{aligned} A &= \{X_0 \exp(\sigma W_T^* - \frac{1}{2}\sigma^2 T) < c\} \\ &= \{W_T^* < \underbrace{\frac{\log(\frac{c}{X_0}) + \frac{1}{2}\sigma^2 T}{\sigma}}_{b}\} \\ &= \{W_T^* < b\}. \end{aligned}$$

We can also express c interms of b.

$$\begin{aligned} b &= \frac{\log(\frac{c}{x_0}) + \frac{1}{2}\sigma^2 T}{\sigma} \\ \Leftrightarrow \ \sigma b &= \log(\frac{c}{x_0}) + \frac{1}{2}\sigma^2 T \\ \Leftrightarrow \ \frac{c}{x_0} &= \exp(\sigma b - \frac{1}{2}\sigma^2 T) \end{aligned}$$

$$c = x_0 \cdot \exp(\sigma b - \frac{1}{2}\sigma^2 T) \tag{4.5.28}$$

Claim: The modified option $H \cdot 1_A$ can be written as a combination of two call options and one binary option as follows

$$H \cdot 1_A = (X_T - K)^+ - (X_T - c)^+ - (c - K)1_{\{X_T > c\}} \tag{4.5.29}$$

Proof:
Let $\omega \in A$ which means that $X_T(\omega) < c$. For the left hand side, we see that $H \cdot 1_A = H$ since $\omega \in A$. For the right hand side,

$$\underbrace{(X_T - K)^+}_{H} - \underbrace{(X_T - c)^+}_{0} - \underbrace{(c - K)1_{\{X_T > c\}}}_{0} = H,$$

so equation (4.5.29) is true.

Now, let $\omega \notin A$ then $X_T(\omega) \geq c$. For the left hand side, $H \cdot 1_A = 0$ since $\omega \notin A$. For the right hand side, we observe that $X_T(\omega) \geq c \Rightarrow X_T(\omega) > K$

$$\begin{aligned} &(X_T - K)^+ - (X_T - c)^+ - (c - K)1_{\{X_T > c\}} = \\ &(X_T - K) - (X_T - c) - (c - K) = 0, \end{aligned}$$

so (4.5.29) is fulfilled. Thus, the claim holds in general.

Using equation (4.5.29), we can now compute the quantile hedging price for the option H, i.e. we compute $V_0 = E^*[H \cdot 1_A]$ by replacing $H \cdot 1_A$ by (4.5.29).

$$\begin{aligned}
E^*[(X_T - K)^+] &= x_0 \cdot \Phi(d_+) - K \cdot \Phi(d_-) \\
E^*[(X_T - c)^+] &= x_0 \cdot \Phi\left(\frac{-\log(\frac{c}{x_0})}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right) - c \cdot \Phi\left(\frac{-\log(\frac{c}{x_0})}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}\right) \\
&\overset{(4.5.28)}{=} x_0 \cdot \Phi\left(\frac{-b+\sigma T}{\sqrt{T}}\right) - c \cdot \Phi\left(\frac{-b}{\sqrt{T}}\right) \\
E^*[(c-K)1_{\{X_T > c\}}] &= (c-K)P^*[X_T > c] \\
&= (c-K)P^*[x_0 \cdot e^{\sigma W_T^* - \frac{1}{2}\sigma^2 T} > c] \\
&= (c-K)P^*\left[W_T^* > \frac{\log(\frac{c}{x_0}) + \frac{1}{2}\sigma^2 T}{\sigma}\right] \\
&\overset{W_T^* \sim \mathcal{N}(0,T)}{=} (c-K)(1 - \Phi\left(\frac{\log(\frac{c}{x_0}) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)) \\
&= (c-K)(1 - \Phi\left(\frac{b}{\sqrt{T}}\right)) = (c-K)\Phi\left(\frac{-b}{\sqrt{T}}\right)
\end{aligned}$$

If we put everything together, we get the quantile hedging price for the call option.

$$\begin{aligned}
V_0 &= x_0 \cdot \Phi(d_+) - K \cdot \Phi(d_-) - x_0 \cdot \Phi\left(\frac{-b+\sigma T}{\sqrt{T}}\right) + c \cdot \Phi\left(\frac{-b}{\sqrt{T}}\right) + (K-c)\Phi\left(\frac{-b}{\sqrt{T}}\right) \\
&= x_0 \cdot \Phi(d_+) - K \cdot \Phi(d_-) - x_0 \cdot \Phi\left(\frac{-b+\sigma T}{\sqrt{T}}\right) + K \cdot \Phi\left(\frac{-b}{\sqrt{T}}\right)
\end{aligned}$$

Given V_0, we can find b from the above equation. Then we can compute the success probability given by $P(W_T^* < b)$.

Equivalently, given the shortfall probability ε, we can find the minimum capital required for the quantile hedging. We know that $A = \{W_T^* < b\}$ and that $W_T^* \sim \mathcal{N}(\frac{m}{\sigma}T, T)$ under P. So

$$P(A) = P(W_T^* < b) = \Phi\left(\frac{b - \frac{m}{\sigma}T}{\sqrt{T}}\right). \tag{4.5.30}$$

Now assume that $P(A) = 1 - \varepsilon$. Then (4.5.30) $\Rightarrow b = \sqrt{T} \cdot \Phi^{-1}(1-\varepsilon) + \frac{m}{\sigma}T$.

Example 1 Consider a call option with T = 0.25, $\sigma = 0.3, \mu = 0.08$, $x_0 = 100$ and $K = 110$, we can compute the values for the rate $\frac{V_0}{H_0}$.

ε	0.01	0.05	0.1
$\frac{V_0}{H_0}$	0.89	0.59	0.34

The table shows that e.g. if we accept a shortfall probability of 5% then we can reduce our initial capital by 41%. These are only some spesific values for ε. See the following graph for the continuum of shortfall probabilities.

4.5.2 Case 2: $m > \sigma^2$

If $m > \sigma^2$ then $X_T^{\frac{m}{\sigma^2}}$ is a convex function.
Because $P(A) < 1$ holds, the success set A is of the form

$$A = \{X_T < c_1\} \cup \{X_T > c_2\} \tag{4.5.31}$$
$$= \{W_T^* < b_1\} \cup \{W_T^* > b_2\}, \tag{4.5.32}$$

where c_1 and c_2 are the two intersection points marked in Figure 4.3. From ca 1, we know that

$$b_i = \frac{\log(\frac{c_i}{x_0}) + \frac{1}{2}\sigma^2 T}{\sigma} \text{ and therefore, } c_i = x_0 \cdot e^{\sigma b_i - \frac{1}{2}\sigma^2 T}. \tag{4.5.33}$$

The constant λ is determined by the condition $E^*[H \cdot 1_A] = V_0$. We know that

$$P(A) = \Phi\left(\frac{b_1 - \frac{m}{\sigma}T}{\sqrt{T}}\right) + \Phi\left(-\frac{b_2 - \frac{m}{\sigma}T}{\sqrt{T}}\right). \tag{4.5.34}$$

Claim: The knock-out option $H \cdot 1_A$ can be written as a sum of three call options and two binary options as follows

$$H \cdot 1_A = (X_T - K)^+ - (X_T - c_1)^+ - (c_1 - K)1_{\{X_T > c_1\}} + (X_T - c_2)^+ + (c_2 - K)1_{\{X_T > c_2\}} \tag{4.5.35}$$

Proof:
Let $\omega \in A$ then $X_T(\omega) < c_1$ or $X_T(\omega) > c_2$. For the left hand side, $H \cdot 1_A = H$ since $\omega \in A$.

For the right hand side, we distinguish two cases: The first case is $X_T(\omega) < c_1$ which implies $X_T(\omega) < c_2$. Hence,

$$\underbrace{(X_T - K)^+}_{H} - \underbrace{(X_T - c_1)^+}_{0} - \underbrace{(c_1 - K)1_{\{X_T > c_1\}}}_{0} + \underbrace{(X_T - c_2)^+}_{0} + \underbrace{(c_2 - K)1_{\{X_T > c_2\}}}_{0} = H,$$

so (4.5.35) is correct. The second case is $X_T(\omega) > c_2$ which implies $X_T(\omega) > c_1$. Therefore,

$$(X_T - K)^+ - (X_T - c_1)^+ - (c_1 - K)1_{\{X_T > c_1\}} + (X_T - c_2)^+ + (c_2 - K)1_{\{X_T > c_2\}}$$
$$= (X_T - K) - (X_T - c_1) - (c_1 - K) + (X_T - c_2) + (c_2 - K) = H.$$

Hence, the claim is true for all $\omega \in A$.

Step 2
Now, let $\omega \notin A$. This means that $X_T(\omega) \geq c_1$ and $X_T(\omega) \leq c_2$. For the left hand side, we see that $H \cdot 1_A = 0$ since $\omega \notin A$.

For the right hand side, we observe that if $X_T(\omega) \geq c_1$, also $X_T(\omega) > K$ holds. Therefore, we have

$$(X_T - K)^+ - (X_T - c_1)^+ - (c - K)1_{\{X_T > c_1\}} + \underbrace{(X_T - c_2)^+}_{0} + \underbrace{(c_2 - K)1_{\{X_T > c_2\}}}_{0}$$
$$= (X_T - K) - (X_T - c_1) - (c_1 - K) + 0 + 0 = 0,$$

so (4.5.35) is also true for $\omega \notin A$.

Using the equation (4.5.35), now we can determine the quantile hedging price for the call option.

$$\begin{aligned}
E^*[(X_T - K)^+] &= x_0 \cdot \Phi(d_+) - K \cdot \Phi(d_-)\\
E^*[(X_T - c_1)^+] &= x_0 \cdot \Phi\left(\frac{-b_1 + \sigma T}{\sqrt{T}}\right) - c_1 \cdot \Phi\left(\frac{-b_1}{\sqrt{T}}\right)\\
E^*[(c_1 - K)1_{\{X_T > c_1\}}] &= (c_1 - K)\Phi\left(\frac{-b_1}{\sqrt{T}}\right)\\
E^*[(X_T - c_2)^+] &= x_0 \cdot \Phi\left(\frac{-b_2 + \sigma T}{\sqrt{T}}\right) - c_2 \cdot \Phi\left(\frac{-b_2}{\sqrt{T}}\right)\\
E^*[(c_2 - K)1_{\{X_T > c_2\}}] &= (c_2 - K)\Phi\left(\frac{-b_2}{\sqrt{T}}\right)
\end{aligned}$$

If we now put these results together, we have the quantile hedging price for the call option, which is equal to

$$\begin{aligned}
V_0 = {} & x_0 \cdot \Phi(d_+) - K \cdot \Phi(d_-) - x_0 \cdot \Phi\left(\frac{-b_1 + \sigma T}{\sqrt{T}}\right) + c_1 \cdot \Phi\left(\frac{-b_1}{\sqrt{T}}\right) - (c_1 - K)\Phi\left(\frac{-b_1}{\sqrt{T}}\right)\\
& + x_0 \cdot \Phi\left(\frac{-b_2 + \sigma T}{\sqrt{T}}\right) - c_2 \cdot \Phi\left(\frac{-b_2}{\sqrt{T}}\right) + (c_2 - K)\Phi\left(\frac{-b_2}{\sqrt{T}}\right) =\\
& = x_0 \cdot \Phi(d_+) - K \cdot \Phi(d_-) - x_0 \cdot \Phi\left(\frac{-b_1 + \sigma T}{\sqrt{T}}\right) + K \cdot \Phi\left(\frac{-b_1}{\sqrt{T}}\right)\\
& + x_0 \cdot \Phi\left(\frac{-b_2 + \sigma T}{\sqrt{T}}\right) - K \cdot \Phi\left(\frac{-b_2}{\sqrt{T}}\right).
\end{aligned}$$

Given V_0 we can not find b_1 and b_2 explicitly but we can find them numerically and then we can compute the success probability.

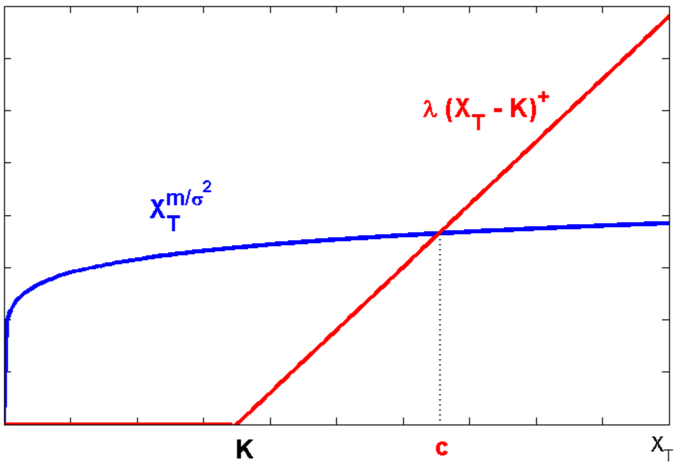

Figure 4.1: $m \leq \sigma^2$

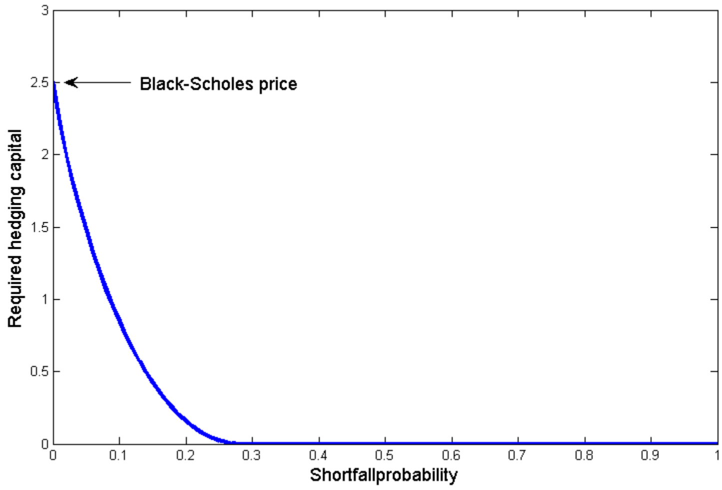

Figure 4.2: $\frac{V_0}{H_0}$

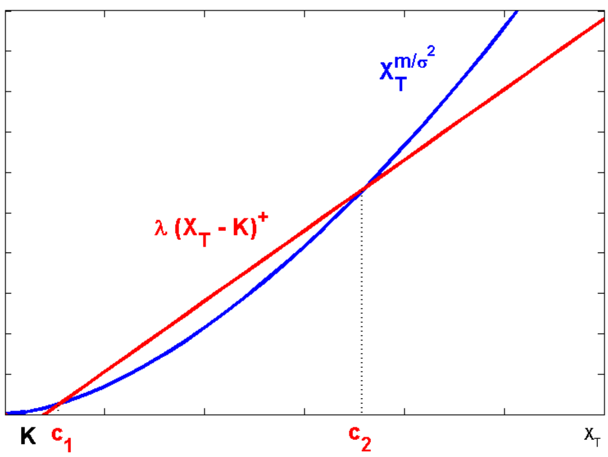

Figure 4.3: $m > \sigma^2$

Chapter 5

Pricing and Hedging

Lecture 10, October 24, 2011

In this chapter, given a financial model $(S_t)_{0\le t\le T}$, $(\Omega, \mathbb{F}, \mathbb{P})$, we consider the question of *pricing and hedging a given financial instrument* H. In general, H is a $\mathcal{F}_T$ measurable random variable. So we would like "infer" the price (or the value) of H given the stock price processes. Implicity, we assume that S is all we know about the market. Simple case would be if

$$H = \sum \alpha_i S_T^i$$

then

$$\text{value}(H) \text{ at time t} = \sum \alpha_i S_t^i$$

But if H is a nonlinear function of (S_T), then the question is more difficult. Moreover, if $\mathcal{F} \underset{\neq}{\supseteq} \mathcal{F}^S$ and if $H \in \mathcal{F}$ but not $\mathcal{F}^S$-mbl, then the question is really probabilistic.

5.1 One step model with finite Ω

In this market,

$$\begin{aligned} \Omega \quad &= \{\omega_1, ..., \omega_K\} \quad \mathcal{F}_0 = \{, \Omega\}, \quad \mathcal{F}_1 = 2^\Omega. \\ S_0 \quad &= (s^1, ...s^d), \quad S_1 = (S^1(\omega), ..., S^d(\omega)). \end{aligned}$$

Set

$$A_{ij} = S^i(\omega_j), \quad i = 1, ..., d, \ j = 1, ..., K.$$

We have proved (NA) is equivalent to the existence of a EMM $\mathbb{Q}$,

$$\mathbb{Q} = (\mathbb{Q}(\omega_1), ..., \mathbb{Q}(\omega_K)) =: (q_1, ..., q_K).$$

It is then easy to verify that S is a $\mathbb{Q}$-martingale iff

$$\begin{aligned} S_0 = \mathbb{E}^{\mathbb{Q}} S_1 \Leftrightarrow s^i = \mathbb{E}^{\mathbb{Q}}(S^i(\cdot)) = \sum_{j=1}^K S^i(\omega_j)\mathbb{Q}(\omega = \omega_j) = \sum A_{ij} q_j, \\ \Leftrightarrow s = Aq \end{aligned}$$

Hence, given $A \in \mathbb{R}^{d\times K}, s \in \mathbb{R}^d$, there is no arbitrage iff there exists $q \in \mathbb{R}^K$ so that

$$s = Aq \text{ and } q \in \Sigma^K \ (\Leftrightarrow 0 \le q_j \le 1, \textstyle\sum q_j = 1).$$

A general $H \in \mathcal{L}^0(\mathcal{F}_1)$ is simply $H = (H(\omega_1), ..., H(\omega_K)) =: (h_1, ..., h_K) \in \mathbb{R}^K$. Then H is a linear combination of S_1 if and only if

$$\begin{aligned} H(\omega) = \sum_{i=1}^{d} \alpha_i S^i(\omega), \forall \omega \ &\Leftrightarrow \ h_j = \sum_{i=1}^{d} \alpha_i S^i(\omega_j) \\ \Leftrightarrow h_j = \sum_{i=1}^{d} \alpha_i A_{ij} \ &\Leftrightarrow \ h = A^T \alpha \end{aligned}$$

So let

$$\mathcal{A} := \{H(\omega_j) = h_j \text{ and } h = A^T \alpha \text{ for some } \alpha \in \mathbb{R}^d\} = \text{Range of}(A^T).$$

Now, if $H \in \mathcal{A}$, then $H = \sum \alpha_i S^i$ and by no arbitrage,

$$\text{value}(H) = \textstyle\sum \alpha_i S^i = \alpha \cdot S.$$

In this case, "hedging" is also very simple. Indeed, instead of buying and holding H, we simply buy and hold α^i shares of the i^{th} stock. We call such claims *attainable* as they can be obtained by trading the underlying stocks.

Definition 5.1 A market is *complete* if all random variables are attainable.

We summarize the above simple discussion in the following.

Lemma 5.2 *In this one step model,*

1. *there is no-arbitrage if and only of there is $q \in \Sigma^K$ so that $s = Aq$,*
2. *it is complete if and only if Range(A^T)$= \mathbb{R}^K$.*

Simple linear algebra yields that

$$\dim(\text{Range}(A^T)) \le \min\{d, K\}.$$

So if the market is complete, then

$$\begin{aligned} K \le \min\{d, K\} &\Leftrightarrow K \le d \\ &\Leftrightarrow \ \# \text{ of sources of risk} \le \# \text{ of tradables}. \end{aligned}$$

If, however, (NA) holds and $d > K$, then there is $\lambda = (\lambda_1, ..., \lambda_d) \in \mathbb{R}^d \backslash \{0\}$ so that

$$A^T \lambda = 0 \ \Rightarrow \lambda \cdot s = \lambda \cdot Aq = q \cdot A^T \lambda = 0.$$

Also

$$(A^T \lambda)_j = \sum_{i=1}^{d} \lambda_i S^i(\omega_j) = \lambda \cdot S_1(\omega_j) = 0.$$

Hence the gain vector

$$\begin{aligned}
&(S_1 - s)\cdot\lambda = 0\ \forall\omega \Leftrightarrow\ \{S_1(\omega_j) - s\}_{j=1,\dots,K} \subset \mathbb{R}^d \ \text{ linearly dependent;}\\
&\Leftrightarrow \exists\ i_0 \text{ s.t. } (S_1^{i_0}(\omega) - s^{i_0}) = \sum_{i\neq i_0} \overline{\alpha}_i (S_1^i(\omega) - S_0^i) \quad \forall\omega;\\
&\Leftrightarrow\ \text{effective tradables is less than } d.
\end{aligned}$$

We make the assumption that $(S_1 - s)$'s are not linearly dependent. Then,

$$\left.\begin{array}{c}(NA) \Rightarrow d \le K\\ (\text{complete}) \Rightarrow d \ge K\end{array}\right\} \Rightarrow\ d = K.$$

We summarize the above discussions in the following,

$$\begin{aligned}
H \text{ attainable } \ \Leftrightarrow\ & H = \sum_i \lambda_i S^i\\
& \Leftrightarrow \text{value}(H) = \sum_i \lambda_i S^i = \sum_i \lambda_i \mathbb{E}^{\mathbb{Q}}(S_1^i)\\
& \Leftrightarrow \text{value}(H) = \mathbb{E}^{\mathbb{Q}}(H)
\end{aligned}$$

Moreover, $q^j = \mathbb{E}^{\mathbb{Q}}(\chi_{\{\omega_j\}})$ is called the *market price of risk.*

5.2 Finite discrete time

We now repeat the one-step argument in finite discrete time.

Definition 5.3 $H \in \mathcal{L}_+^0(\mathcal{F}_T)$ is *attainable* if there exists a self-financing, admissible strategy (V_0, ϑ) satisfying $V_T(\vartheta) = H, \mathbb{P}$-a.s. This strategy is called the *hedging portfolio.*

A market $((S_t)_{t=0,\dots,T}, \mathbb{F}, P)$ is called *complete* if every $H \in \mathcal{L}_+^0(\mathcal{F}_T)$ is attainable.

We summarize these definitions in the following result.

Proposition 5.4 *Suppose $\mathcal{F}_0$ is trivial. Then, the following are equivalent.*

(a) $(S, \mathbb{F}, \mathbb{P})$ is complete,

(b) For every $H \in \mathcal{L}_+^0(\mathcal{F}_T)$, there exists an $\mathbb{F}$-predictable, S-integrable, $\mathbb{R}^d$-valued ϑ and a constant H_0 such that

$$\begin{aligned}
H &= H_0 + \textstyle\int_0^T \vartheta_u dS_u = H_0 + \displaystyle\sum_{i=1}^T \vartheta_i (S_i - S_{i-1}),\ \mathbb{P} - a.s.,\\
G_t(\vartheta) &= \textstyle\int_0^t \vartheta_u dS_u \ge -c(\vartheta)\ \mathbb{P} - a.s.,
\end{aligned}$$

where $c(\vartheta)$ is a deterministic constant.

Proof. There is nothing to prove. All above statements follow directly from definitions. ∎

Recall that $\mathcal{P}^e := \{\mathbb{Q} \approx \mathbb{P},\ S \text{ is a } \mathbb{Q}\text{-martingale}\}$. Then, in finite discrete time $NA \Leftrightarrow \mathcal{P}^e \neq \emptyset$

Theorem 5.5 *Suppose $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and (NA) holds. For $H \in \mathcal{L}^0_+(\mathcal{F}_T)$, the following are equivalent,*

(a) H is attainable;

(b) $\sup\limits_{\mathbb{Q}\in\mathcal{P}^e} \mathbb{E}^{\mathbb{Q}}(H) < \infty$ and the supremum is attained,

(c) $\mathbb{E}^{\mathbb{Q}}(H) = \mathbb{E}^{\mathbb{Q}'}(H) < \infty$ for every $\mathbb{Q}, \mathbb{Q}' \in \mathcal{P}^e$.

This is an intuitive result. Indeed, if H is attained then its value is given by the initial value of the hedging portfolio. So the price of an attainable claim is well defined. On the other hand for any martingale measure $\mathbb{Q}$, the expected value $\mathbb{E}^{\mathbb{Q}}(H)$ is a possible price. However, since the price is well defined, one expects that all expected values are the same for an attainable claim.

Proof. $(a) \Rightarrow (c)$: Since H is attainable,

$$H = H_0 + \int^T \vartheta_u dS_u \text{ and } G(\vartheta) = \int^T \vartheta_u dS_u \geq -c(\vartheta).$$

Then, for any $\mathbb{Q} \in \mathcal{P}^e$, $G(\vartheta)$ is a $\mathbb{Q}$-martingale and

$$\mathbb{E}^{\mathbb{Q}}(H) = H_0,\ \forall \mathbb{Q} \in \mathcal{P}^e.$$

$(c) \Rightarrow (b)$ is clear.

$(b) \Rightarrow (a)$. Define

$$U_k := \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{P}^e} \ \mathbb{E}^{\mathbb{Q}}(H|\mathcal{F}_k).$$

(We give the definition of the essential supremum after the proof.) Since $H \geq 0$, $U_k \geq 0$ for all k. Also, since $\mathcal{F}_0$ is trivial, the essential supremum at $k = 0$ is simply the supremum and by hypothesis (b), there exists $\mathbb{Q}^* \in \mathcal{P}^e$ so that

$$U_0 = \sup_{\mathbb{Q}\in\mathcal{P}^e} \mathbb{E}^{\mathbb{Q}}(H) = \mathbb{E}^{\mathbb{Q}^*}(H) < \infty.$$

We now claim that $\{U_k\}_{k=0,\ldots,T}$ is a $\mathbb{Q}$-supermartingale for each $\mathbb{Q} \in \mathcal{P}^e$. We postpone the proof of this claim and continue with the original proof. Then, the uniform decomposition (or optional decomposition) theorem [1] (also it will be stated and proved later), there are an adapted, non-decreasing process C with $C_o = 0$ and a predictable, S-integrable, $\mathbb{R}^d$-valued process ϑ such that

$$U_k = U_0 + \sum_{i=1}^{k} \vartheta_i(S_i - S_{i-1}) - C_k.$$

[1] In general, a supermartingale can be decomposed into a martingale and a non-decreasing process,i.e., $U = M - C$ and C is predictable. But then M may not be a stochastic integral. This is the case if M is a P_0-$\mathcal{F}^B$ martingale but here we have completeness. So the fact that the market is complete is crucial even in discrete time.

Hence,

$$U_0 + \int \vartheta dS = U + C \geq 0,$$

and (U_0, ϑ) is admissible. Moreover, $U_T = H$ as H is $\mathcal{F}_T$-measurable. So H would be attainable if we can show that $C \equiv 0$ $\mathbb{P}$-almost surely. This is equivalent (since C is non-decreasing) to $C_T = 0$ $\mathbb{P}$-almost surely. We directly calculate that

$$\mathbb{E}^{\mathbb{Q}}(U_T + C) = \mathbb{E}^{\mathbb{Q}}(U_0 + \int \vartheta dS) \leq U_0.$$

(Here we use Ansel-Stricker to conclude that any stochastic integral bounded from below is a supermartingale. With a bit more work it can be shown to be a martingale but we do not need it here.) Hence,

$$\begin{aligned} U_0 &\geq \sup_{\mathbb{Q} \in \mathcal{P}^e} \mathbb{E}^{\mathbb{Q}}(U_T + C_T) \geq \mathbb{E}^{\mathbb{Q}^*}(U_T + C_T) = \mathbb{E}^{\mathbb{Q}^*}(U_T) + \mathbb{E}^{\mathbb{Q}^*}(C_T) \\ &= U_0 + \mathbb{E}^{\mathbb{Q}^*}(C_T) \\ &\Rightarrow \mathbb{E}^{\mathbb{Q}^*}(C_T) \leq 0 \;\Rightarrow\; C_T = 0 \quad \mathbb{Q}^* - a.s. \Rightarrow C_T = 0,\; \mathbb{P} - a.s. \end{aligned}$$

■

In the above proof, we used the essential supremum and the uniform Dob-Meyer decomposition of super-martingales. We discuss these below.

Lecture 11, October 27, 2011

Essential supremum.

The question is as follows. We are given a family of random variables $\{Y^\lambda\}_{\lambda \in \Lambda}$; i.e., for each $\lambda \in \Lambda$,

$$Y^\lambda : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}).$$

We are also given measure μ on $(\Omega, \mathcal{F})$. Then, *the essential supremum* of $(Y^\lambda)_{\lambda \in \Lambda}$ is an $\mathcal{F}$-measurable function Y satisfying,

i Y is of $\mathcal{F}$-measurable,

ii $Y \geq Y^\lambda$ μ-a.s., $\forall \lambda \in \Lambda$,

iii if Z is a $\mathcal{F}$-mbl and $Z \geq Y^\lambda$ μ-a.s., for every $\lambda \in \Lambda$, then

$$Z \geq Y,\; \mu - a.s.$$

The existence of the essential supremum is proved below under the assumption of upward directedness. Indeed, we say that the family $(Y^\lambda)_{\lambda \in \Lambda}$ is *upward directed* if for any $\lambda_1, \lambda_2 \in \Lambda$ there exists $\lambda \in \Lambda$ so that

$$Y^{\lambda_1} \vee Y^{\lambda_2} \leq Y^\lambda,\; \mu - a.s.$$

Lemma 5.6 *Suppose $(\Omega, \mathcal{F}, \mu)$ is σ-finite and that the family $\{Y^\lambda\}_{\lambda\in\Lambda}$ is upward directed. Then,*

$$Y = \operatorname*{\mathit{ess\,sup}}_{\lambda\in\Lambda} \; Y^\lambda$$

exists and is unique upto μ-a.s., equivalences. Moreover, there is a sequence $Y_n := Y^{\lambda_n}$ so that

$$Y_{n+1} \geq Y_n, \;\; \mu - a.s., \;\; and \;\; \lim_{n\to\infty} Y_n = \sup_n Y_n = Y, \;\; \mu - a.s.$$

Proof. Choose any continuous, strictly increasing function $\varphi : \mathbb{R} \to (-1, 1)$ (i.e. $\varphi(x) = x/1 + |x|$). Set

$$\alpha := \sup_{\lambda\in\Lambda} \int \varphi(Y^\lambda) d\mu \;\; \leq 1.$$

Then, there exists a sequence $\hat{\lambda}_n \in \Lambda$ so that

$$\alpha = \lim \int \varphi(Y^{\hat{\lambda}_n}) d\mu.$$

By upward-directedness, for each n there are λ_n so that $\lambda_1 = \hat{\lambda}_1$ and

$$Y_n := Y^{\lambda_n} \geq \max\{Y^{\hat{\lambda}_1}, \ldots, Y^{\hat{\lambda}_n}\} \vee Y_{n-1}, \;\; n = 2, 3, ...$$

Then $Y_{n+1} \geq Y_n$ and

$$\alpha = \sup_n \int \varphi(Y_n) d\mu.$$

Set

$$Y := \sup_n Y_n.$$

We claim that Y is an essential supremum. Indeed, if $\mu(Y^\lambda - Y \geq \delta) \geq 0$ for some $\lambda \in \Lambda$, then choose a sequence $\overline{Y}_n$ by $\overline{Y}_1 \geq Y_1 \vee Y^\lambda$,

$$\overline{Y}_n \geq (Y_n \vee Y^\lambda) \vee \overline{Y}_{n-1}, \quad n = 2, ...$$

Then on $A := \{Y^\lambda - Y \geq \delta\}$, $\overline{Y}_n = Y^\lambda$. Hence,

$$\varphi(\overline{Y}_n) \geq \varphi(Y_n)\chi_{A^c} + \varphi(Y + \delta)\chi_A$$

and

$$\lim \int \varphi(\overline{Y}_n) d\mu \geq \alpha + \int (\varphi(Y + \delta) - \varphi(Y))\chi_A d\mu > \alpha.$$

Since this clearly contradicts with the fact that α is the supremum, we conclude that $Y \geq Y^\lambda$ μ-a.s., for every $\lambda \in \Lambda$.

If Z is another $\mathcal{F}$-mbl function that is $Z \geq Y^\lambda$ μ-a.s. for $\lambda \in \Lambda$, then

$$\begin{aligned} &Z \geq Y^n \;\; \mu - \text{a.s. } \forall n \\ &\Rightarrow Z \geq \;\sup\, Y^n = Y \quad \mu - \text{a.s.} \end{aligned}$$

Uniqueness: Suppose Y, Z both satisfy the conditions of an essential supremum. Then, $Y \geq Z$ and $Z \geq Y$ μ-a.s.. Hence $Y = Z$ μ-a.s. ∎

Remark 5.7 In general, $\operatorname{ess\,sup}_{\lambda\in\Lambda} Y^\lambda$ exists even if Y^λ is not upward directed. But in that we can not construct an increasing sequence of Y_n's, which is extremely useful in applications.

Super-martingale property of the essential supremum

This claim is postponed in the above proof which we complete now.

We start with a computational Lemma that will be used several times in the proof. We use the notation with any measure $\mathbb{P}$,

$$\mathbb{E}_t^{\mathbb{P}}(H) := \mathbb{E}^{\mathbb{P}}(H|\mathcal{F}_s).$$

Lemma 5.8 *For two measures $\mathbb{Q}$ and $\mathbb{K}$ on $(\Omega, \mathcal{F})$ and $A \in \mathcal{F}_t$, define*

$$\mathbb{P}(B) := \mathbb{E}^{\mathbb{Q}}[\mathbb{Q}(B|\mathcal{F}_t)\chi_A + \mathbb{K}(B|\mathcal{F}_t)\chi_{A^c}],\ B \in \mathcal{F}_T.$$

Then, $\mathbb{P}$ is a probability measure satisfying,

1. $\mathbb{P}(B) = \mathbb{Q}(B),\ \forall B \in \mathcal{F}_t$.
2. $\mathbb{E}^{\mathbb{P}}(Z) = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}_t^{\mathbb{Q}}(Z)\chi_A + \mathbb{E}_t^{\mathbb{K}}(Z)\chi_{A^c}]$, *for every $Z \in \mathcal{L}^\infty$.*
3. $\mathbb{E}_s^{\mathbb{P}}(Z) = \mathbb{E}_s^{\mathbb{Q}}[\mathbb{E}_t^{\mathbb{Q}}(Z)\chi_A + \mathbb{E}_t^{\mathbb{K}}(Z)\chi_{A^c}]$, *for every $Z \in \mathcal{L}^\infty$ and $s \le t$.*
4. $\mathbb{E}_t^{\mathbb{P}}(Z) = \mathbb{E}_t^{\mathbb{Q}}(Z)\chi_A + \mathbb{E}_t^{\mathbb{K}}(Z)\chi_{A^c}$, *for every $Z \in \mathcal{L}^\infty$.*
5. $\mathbb{E}_s^{\mathbb{P}}(Z) = \mathbb{E}_s^{\mathbb{Q}}(Z)\chi_A + \mathbb{E}_s^{\mathbb{K}}(Z)\chi_{A^c}$, *for every $Z \in \mathcal{L}^\infty$ and $s \le t$,*
6. *If S is a martingale under both $\mathbb{Q}$ and $\mathbb{K}$ and it is also a $\mathbb{P}$ martingale.*

Proof. The item one is clear from the definition.

2. By definition, it holds for all Z in the form $Z = \chi_B$. the general case follows from a direct approximation argument.

3. We need to check that

$$\mathbb{E}^{\mathbb{P}}(ZY) = \mathbb{E}^{\mathbb{P}}\left[Y\ \mathbb{E}_s^{\mathbb{Q}}\left(\mathbb{E}_t^{\mathbb{Q}}(Z)\chi_A + \mathbb{E}_t^{\mathbb{K}}(Z)\chi_{A^c}\right)\right],$$

for every $Y \in \mathcal{L}^\infty(\mathcal{F}_s)$. We calculate directly that

$$\mathbb{E}^{\mathbb{P}}(ZY) = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}_t^{\mathbb{Q}}(ZY)\chi_A + \mathbb{E}_t^{\mathbb{K}}(ZY)\chi_{A^c}].$$

Since Y is $\mathcal{F}_s$ measurable and also since $\mathbb{P} = \mathbb{Q}$ on $\mathcal{F}_t$ and $\mathcal{F}_s \subset \mathcal{F}_t$, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}(ZY) &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}_s^{\mathbb{Q}}\left(\mathbb{E}_t^{\mathbb{Q}}(ZY)\chi_A + \mathbb{E}_t^{\mathbb{K}}(ZY)\chi_{A^c}\right)\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[Y\ \mathbb{E}_s^{\mathbb{Q}}\left(\mathbb{E}_t^{\mathbb{Q}}(Z)\chi_A + \mathbb{E}_t^{\mathbb{K}}(Z)\chi_{A^c}\right)\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[Y\ \mathbb{E}_s^{\mathbb{Q}}\left(\mathbb{E}_t^{\mathbb{Q}}(Z)\chi_A + \mathbb{E}_t^{\mathbb{K}}(Z)\chi_{A^c}\right)\right].
\end{aligned}$$

4 This follows immediately from the previous step.

5. Again let $Y \in \mathcal{L}^\infty(\mathcal{F}_S)$. Then using the previous step,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}(ZY) &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}_t(ZY)\chi_A + \mathbb{E}^{\mathbb{K}}_t(ZY)\chi_{A^c}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}_t\left(\mathbb{E}^{\mathbb{Q}}_s(ZY)\right)\chi_A + \mathbb{E}^{\mathbb{K}}_t\left(\mathbb{E}^{\mathbb{K}}_s(ZY)\right)\chi_{A^c}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}_t\left(E^{\mathbb{Q}}_s(ZY)\chi_A + \mathbb{E}^{\mathbb{K}}_s(ZY)\chi_{A^c}\right)\chi_A\right. \\
&\quad \left. + \mathbb{E}^{\mathbb{K}}_t\left(E^{\mathbb{Q}}_s(ZY)\chi_A + \mathbb{E}^{\mathbb{K}}_s(ZY)\chi_{A^c}\right)\chi_{A^c}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}_t\left(E^{\mathbb{Q}}_s(ZY)\chi_A + \mathbb{E}^{\mathbb{K}}_s(ZY)\chi_{A^c}\right)\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[Y\mathbb{E}^{\mathbb{P}}_t\left(E^{\mathbb{Q}}_s(Z)\chi_A + \mathbb{E}^{\mathbb{K}}_s(Z)\chi_{A^c}\right)\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[Y\left(E^{\mathbb{Q}}_s(Z)\chi_A + \mathbb{E}^{\mathbb{K}}_s(Z)\chi_{A^c}\right)\right].
\end{aligned}$$

6. This follows from the previous steps.

■

We now prove the supermartingale property used in the proof of attainable claims. We restate the result and prove it.

Lemma 5.9 *Let $\mathcal{P}^e$ be the set of all equivalent martingale measures. and $H \in \mathcal{L}^0_+(\mathcal{F}_T)$. Set*

$$U_k := \textit{ess sup}_{\ \mathbb{Q}\in\mathcal{P}^e}\ \mathbb{E}^{\mathbb{Q}}[\mathrm{H}].$$

Assume that U_0 is finite. Then, U is a $\mathbb{Q}$ supermartingale for every $\mathbb{Q} \in \mathcal{P}^e$.

Proof. We complete it in several steps.

1. First we prove that the family $\{\mathbb{E}^{\mathbb{Q}}(H|\mathcal{F}_t)\ :\ \mathbb{Q} \in \mathcal{P}^e\}$ is upward-directed. Indeed let $\mathbb{Q}, \mathbb{P} \in \mathcal{P}^e$ and set

$$A := \{\mathbb{E}^{\mathbb{Q}}(H|\mathcal{F}_t) \geq \mathbb{E}^{\mathbb{P}}(H|\mathcal{F}_t)\} \in \mathcal{F}_t.$$

Define $\hat{\mathbb{Q}}$ on $\mathcal{F}_T$ by

$$\hat{\mathbb{Q}}(B) := \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}(\chi_B|\mathcal{F}_t)\chi_A + \mathbb{E}^{\mathbb{P}}(\chi_B|\mathcal{F}_t))\chi_{A^c}], \quad \forall B \in \mathcal{F}_T.$$

By the previous lemma, $\hat{\mathbb{Q}} \in \mathcal{P}^e$. Moreover,

$$\begin{aligned}
\mathbb{E}^{\hat{\mathbb{Q}}}(H|\mathcal{F}_t) &= \mathbb{E}^{\mathbb{Q}}[H|\mathcal{F}_t]\chi_A + \mathbb{E}^{\mathbb{P}}(H|\mathcal{F}_t)\chi_{A^c} \\
&= \max\{\mathbb{E}^{\mathbb{Q}}(H|\mathcal{F}_t), \mathbb{E}^{\mathbb{P}}(H|\mathcal{F}_t)\}.
\end{aligned}$$

2. Therefore essential supremum exists and is given as a limit. Hence, for any $\mathbb{Q} \in \mathcal{P}^e$ and k,

$$\mathbb{E}^{\mathbb{Q}}(U_{k+1}|\mathcal{F}_k) = \lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}}(\mathbb{E}^{\mathbb{Q}_n}(H|\mathcal{F}_{k+1})|\mathcal{F}_k).$$

We claim that there exists a sequence $\{\overline{\mathbb{Q}}_n\} \subset \mathcal{P}^e$ such that

$$\mathbb{E}^{\mathbb{Q}}(\mathbb{E}^{\mathbb{Q}_n}(Z|\mathcal{F}_{k+1})|\mathcal{F}_k) = \mathbb{E}^{\overline{\mathbb{Q}}_n}(Z|\mathcal{F}_k), \quad \forall Z \in L^\infty(\mathcal{F}_T). \tag{5.2.1}$$

When this sequence constructed, then

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[U_{k+1}|\mathcal{F}_k] &= \lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}_n}[H|\mathcal{F}_{k+1}]|\mathcal{F}_k] \\ &= \lim_{n\to\infty} \mathbb{E}^{\overline{\mathbb{Q}}_n}[H|\mathcal{F}_k] \\ &\le \operatorname*{ess\,sup}_{\tilde{\mathbb{Q}}\in\mathcal{P}^e} \mathbb{E}^{\tilde{\mathbb{Q}}}[H|\mathcal{F}_k] = U_k\end{aligned}$$

Hence, (U_n) is a $\mathbb{Q}$-supermartingale.

3. In this step we prove the claim (5.2.1). Indeed, for given $\mathbb{Q}, \mathbb{K} \in \mathcal{P}^e$ we define $\hat{\mathbb{Q}}$ by

$$\hat{\mathbb{Q}}(B) = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{K}}[\chi_B|\mathcal{F}_k]] \;\Leftrightarrow \mathbb{E}^{\hat{\mathbb{Q}}}[Z] = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{P}[Z|\mathcal{F}_{k_0}]].$$

We again use the previous lame but A being equal to the empty set. Hence, $\tilde{\mathbb{Q}} \in \mathcal{P}^e$. ■

So the important steps in the discrete time problem are,

1. definition of essential supremum and it is not specific to discrete time;
2. supermartingale property of the essential supremum and it also generalizes easily;
3. optional decomposition, this also generalizes but no easily.

We will do optional decomposition in the discrete time in its full generality.

Lecture 12, October 31, 2011

5.3 Optional (or uniform Doob) decomposition

We start with the usual decomposition in discrete time and point out the differences.

Lemma 5.10 *Suppose that and adapted process* $(U_k)_{k=0,\dots,T}$ *is a* Q*-supermartingale. Then*

$$U_k = U_0 + M_k - C_k$$

where C_k *is a predictable, non-decreasing process with* $C_0 = 0$ *and* M *is a* Q*-martingale.*

Proof. Set,

$$\begin{aligned} C_1 &:= U_0 - \mathbb{E}[U_1] \geq 0, \\ M_1 &= U_1 + C_1,\ V_0 = U_0. \end{aligned}$$

Then,

$$\mathbb{E}[V_1] = V_0,$$

hence is a one-step martingale. We proceed as this by setting

$$\begin{aligned} C_{k+1} &= C_k + U_k - \mathbb{E}^{\mathbb{Q}}(U_{k+1}|\mathcal{F}_k) \\ M_{k+1} &= U_{k+1} + C_{k+1},\ k = 1, 2, ..., T-1. \end{aligned}$$

Then, C has the desired properties, and

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}(M_{k+1}|\mathcal{F}_k) &= \mathbb{E}^{\mathbb{Q}}(U_{k+1}|\mathcal{F}_k) + [C_k + U_k - \mathbb{E}^{\mathbb{Q}}(U_{k+1}|\mathcal{F}_k)] \\ &= C_k + U_k \\ &= M_k. \end{aligned}$$

∎

However, in general M may not be a stochastic integral. Simply take $T = 1$, $\Omega = [-1, 1]$, $\mathbb{P}$ =uniform and $S_1(\omega) = \omega$, $S_0 = 0$. Then, S is a $\mathbb{P}$-martingale. Also, any process $M(\omega)$ is a $\mathbb{P}$-martingale if and only if,

$$0 = \int_{-1}^{1} M(x) dx$$

and this does not necessarily imply that $M(x)$ is linear in x, which would mean integral representation. But in this model there are many equivalent martingale measures. Indeed $\mathbb{Q}$ is an EMM iff $(d\mathbb{Q}/dx)(x) := Z^{\mathbb{Q}}(x)$ satisfies

$$\int_{-1}^{1} Z^{\mathbb{Q}}(x) x dx = 0,\ \int_{-1}^{1} Z^{\mathbb{Q}}(x) dx = 1,\ Z^{\mathbb{Q}}(x) \geq 0.$$

If M is a martingale with respect to all EMM's, that is a very strong condition: $\int_{-1}^{1} M(x) Z^{\mathbb{Q}}(x) dx = 0$. Then, one may conclude that M is linear.

Theorem 5.11 (Theorem 7.5, in F&S) *For an adapted, non-negative process U (in discrete time), the following are equivalent,*

(a) U is a $\mathcal{P}^e$-supermartingale,

(b) There exists an adapted non-decreasing process C with $C_0 = 0$ and a predictable process ϑ so that

$$U_t = U_0 + \sum_{i=1}^{t} \vartheta_i (S_i - S_{i-1}) - C_t, \ \mathbb{P} - a.s., \ \forall t = 1, ..., T.$$

Proof. $(b) \Rightarrow (a)$ is easy.

$(a) \Rightarrow (b)$: We need to show that for each t,

$$U_t - U_{t-1} = \vartheta_t \cdot (S_t - S_{t-1}) - R_t, \tag{5.3.2}$$

for some $\vartheta \in \mathcal{L}^0(\mathcal{F}_{t-1})$ and $R \in \mathcal{L}^0_+(\mathcal{F}_t)$. As in the no-arbitrage theorem, we set

$$K_t := \{\vartheta \cdot Y_t \ : \ \vartheta \in \mathcal{L}^0(\mathcal{F}_{t-1})\} \text{ and } Y_t = S_t - S_{t-1}.$$

Then, (5.3.2) is equivalent to

$$U_t - U_{t-1} \in K_t - \mathcal{L}^0_+(\mathcal{F}_t).$$

Without loss of generality, we assume $\mathbb{P} \in \mathcal{P}^e$. Then, $U_t - U_{t-1} \in \mathcal{L}^1(\mathcal{F}_t)$ and hence, we need to show that

$$U_t - U_{t-1} \in \mathcal{Z} := (K_t - \mathcal{L}^0_+(\mathcal{F}_t)) \cap \mathcal{L}^1(\mathcal{F}_t).$$

Suppose, for a contraposition argument, that

$$U_t - U_{t-1} \notin \mathcal{Z}.$$

Since $\mathcal{P}^e \neq \emptyset$, no-arbitrage holds. Therefore, $\mathcal{Z}$ is closed. By the separation argument of Hahn-Banach, there exists $Z \in \mathcal{L}^\infty(\mathcal{F}_t)$ so that

$$\alpha := \sup_{W \in \mathcal{Z}} \mathbb{E}^{\mathbb{P}}[ZW] < \mathbb{E}^{\mathbb{P}}[Z(U_t - U_{t-1})] := \delta < \infty.$$

Since $\mathcal{Z}$ is a cone, $\alpha = 0$. We now proceed as in Lemma 1.5.7 of F & S, that implies that $Z \geq 0$, $\mathbb{P}$-a.s. and

$$\mathbb{E}^{\mathbb{P}}(ZY|\mathcal{F}_{t-1}) = \mathbb{E}^{\mathbb{P}}(Z(S_t - S_{t-1})|\mathcal{F}_{t-1}) = 0, \ \mathbb{P} - a.s.$$

Now, for $0 < \epsilon \ll 1$, set $Z^\epsilon := Z + \epsilon$. Then, for $W \in \mathcal{Z}$

$$W = \vartheta Y \Rightarrow \mathbb{E}^{\mathbb{P}}(Z^\epsilon W) = \mathbb{E}^{\mathbb{P}}(ZW) + \epsilon \mathbb{E}^{\mathbb{P}}(W) \leq \mathbb{E}^{\mathbb{P}}(ZW) \leq 0.$$

Also,

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}([Z^\epsilon(U_t - U_{t-1})] &= \mathbb{E}^{\mathbb{P}}[Z(U_t - U_{t-1})] + \epsilon \mathbb{E}^{\mathbb{P}}(U_t - U_{t-1}) \\ &= \delta + \epsilon \mathbb{E}^{\mathbb{P}}(U_t - U_{t-1}) \\ &\leq \delta/2, \end{aligned}$$

provided that $\epsilon > 0$ is sufficiently small. So without loss of generality, we may assume that $Z \geq \epsilon$, $\mathbb{P}$-a.s. for some $\epsilon > 0$. Set

$$Z_{t-1} := \mathbb{E}^{\mathbb{P}}[Z|\mathcal{F}_{t-1}], \text{ and } \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} := \frac{Z}{Z_{t-1}}.$$

Note that t is a fixed time point. We claim that $\tilde{\mathbb{P}} \in \mathcal{P}^e$. Indeed,

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_k} = \varphi_k := \mathbb{E}^{\mathbb{P}}\left[\frac{Z}{Z_{t-1}}|\mathcal{F}_k\right].$$

Since Z is $\mathcal{F}_t$-measurable, for $k \geq t$,

$$\varphi_k = \varphi_t = \frac{Z}{Z_{t-1}},$$

and for $k < t$

$$\varphi_k = \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\frac{Z}{Z_{t-1}}|\mathcal{F}_{t-1}\right]|.\mathcal{F}_k\right] = 1.$$

Moreover,

$$J := \mathbb{E}^{\tilde{\mathbb{P}}}[S_k - S_{k-1}|\mathcal{F}_{k-1}] = \mathbb{E}^{\mathbb{P}}\left[\frac{Z}{Z_{t-1}}(S_k - S_{k-1})|\mathcal{F}_{k-1}\right].$$

For $k > t$, (Z/Z_{t-1}) is $\mathcal{F}_t$ measurable and $\mathcal{F}_t \subset \mathcal{F}_{k-1}$. Hence,

$$J = \frac{Z}{Z_{t-1}}\mathbb{E}^{\mathbb{P}}(S_k - S_{k-1}|\mathcal{F}_{k-1}) = 0.$$

For $k < t$

$$\begin{aligned} J &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left(\frac{Z}{Z_{t-1}}(S_k - S_{k-1})|\mathcal{F}_{t-1}\right)|\mathcal{F}_{k-1}\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[(S_k - S_{k-1})\mathbb{E}^{\mathbb{P}}\left(\frac{Z}{Z_{t-1}}|\mathcal{F}_{t-1}\right)|\mathcal{F}_{k-1}\right] = 0. \end{aligned}$$

If $k = t$

$$J = \frac{1}{Z_{t-1}}\mathbb{E}^{\mathbb{P}}[(S_t - S_{t-1})Z|\mathcal{F}_{t-1}] = 0,$$

by the construction of Z. Hence $\tilde{\mathbb{P}} \in \mathcal{P}^e$. Since $\tilde{\mathbb{P}} \in \mathcal{P}^e$, U is a $\tilde{\mathbb{P}}$-martingale and

$$\begin{aligned} &0 \geq \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{E}^{\tilde{\mathbb{P}}}(U_t - U_{t-1}|\mathcal{F}_{t-1})Z_{t-1}] = \mathbb{E}^{\tilde{\mathbb{P}}}[(U_t - U_{t-1})Z_{t-1}] \\ &\quad = \mathbb{E}^{\mathbb{P}}[(U_t - U_{t-1})Z] = \delta. \end{aligned}$$

This is a contradiction.

$\blacksquare$

We have the following simple corollary to the characterization of attainable claims.

Theorem 5.12 *Assume (NA). Then $(S, \mathbb{F})$ is complete if and only if $\mathcal{P}^e$ is a singleton.*

Proof. 2) $\Rightarrow$ 1) : For any $H \in \mathcal{L}^0_+(\mathcal{F}_T)$, the map $\mathbb{E}^{\mathbb{Q}}(H)$ as $\mathbb{Q} \in \mathcal{P}^e$ is trivially constant and our previous result applies.

1) $\Rightarrow$ 2) : Fix $A \in \mathcal{F}_T$. Since $H := \chi_A$ is attainable, there is an admissible portfolio (V_0, ϑ) so that

$$\chi_A = V_0 + \int_0^T \vartheta_u dS_u$$

where V_0 is a real number ($\mathcal{F}_0$ is trivial) and by admissibility there is c so that

$$G(\vartheta) = \int \vartheta dS \geq -c.$$

Then, for any $\mathbb{Q} \in \mathcal{P}^e$,

$$\mathbb{Q}(A) = \mathbb{E}^{\mathbb{Q}} \chi_A = V_0, \quad \forall \mathbb{Q} \in \mathbb{P}.$$

■

So in finite discrete time

$$\begin{aligned} (NA) \ &\Leftrightarrow \ \mathcal{P}^e \neq \emptyset, \\ (NA) + \text{completeness} \ &\Leftrightarrow \ \mathcal{P}^e = \{\mathbb{P}^*\}. \end{aligned}$$

Continuous time:

The continuos time version was proved by D. Kramkov in 1996. In the case of Brownian filtration proof is easier and was known. Kramov's proof is, however, more general. The structure of the proof is similar to the discrete time version and uses may results and ideas from the fundamental paper of Delbaen & Schachermayer on no-arbitrage. Here we only give the statement.

Theorem 5.13 (Kramkov, PTRF 105,459-479,1996.) *Let $(V_t)_{t \geq 0}$ be a non-negative process. Then, V is a supermartingale for every equivalent local martingale measure, if and only f there are a S-integrable predictable process H and an adapted process C so that*

$$V_t = V_0 + \int_0^t H_u dS_u - C_t, \quad \forall t \geq, \mathbb{P}.a.s.$$

Chapter 6

Super-replication

Lecture 13, November 3, 2011, Thursday

As before we assume that $(\Omega, \mathbb{F}, \mathbb{P})$ and S on $[0,T]$ are given and $\mathcal{F}_0$ is trivial. We set $\mathcal{P}^e$ to be the set of all ELMM's and we assume $\mathcal{P}^e \neq \emptyset$. Recall

$$\mathcal{P} \neq \emptyset \Rightarrow NFLVR$$
$$\mathcal{P} \neq \emptyset \Leftarrow NFLVR \text{ and } S \geq 0.$$

6.1 Seller's Price

We fix a payoff $H \in \mathcal{L}^0_+(\mathcal{F}_T)$. The smallest cost to the sells which carries zero risk is defined by

$$\begin{aligned}\Pi_{seller}(H) &:= \inf\{V_0 \in \mathbb{R} \mid V_0 + \textstyle\int^T \vartheta_u dS_u \geq H \quad P-a.s. \text{ for some } \vartheta \in \Theta_{adm}\} \\ &= \inf\{V_0 \in \mathbb{R} \mid H - V_0 \in G_T(\Theta_{adm}) - L^0_+\}.\end{aligned}$$

In the above definition, proving that infimum is indeed a minimum is a difficult question.

Lemma 6.1 *For any $H \in \mathcal{L}^0_+(\mathcal{F}_T)$,*

$$\Pi_{seller}(H) \geq \sup_{\mathbb{Q} \in \mathcal{P}^e} \mathbb{E}^{\mathbb{Q}}[H].$$

Proof. Suppose $V_0 + \int^T \vartheta_u dS_u \geq H$ $\mathbb{P}$-a.s. Since $\vartheta \in \Theta_{adm}$, for any $\mathbb{Q} \in \mathcal{P}^e$, $\int^T \vartheta_u dS_u$ is a local martingale. Hence, we have

$$\mathbb{E}^{\mathbb{Q}}[H] \leq V_0 + \mathbb{E}^{\mathbb{Q}}[G_T(\vartheta)] \leq V_0.$$

∎

For $t \in [0,T]$ set

$$U_t := \operatorname*{ess\,sup}_{\mathbb{Q} \in \mathcal{P}^e} \mathbb{E}^{\mathbb{Q}}(H|\mathcal{F}_t) \ \Rightarrow \ U_0 = \sup_{\mathbb{Q} \in \mathcal{P}^\rceil} \mathbb{E}^{\mathbb{Q}}(H).$$

Proposition 6.2 *If $U_0 < \infty$, then U is a $\mathcal{P}^e$-supermartingale.*

Proof. Fix $\mathbb{Q} \in \mathbb{P}$ and set,

$$\begin{aligned} \mathcal{Z}_t : &= \{Z \ : \ Z = \frac{dR}{d\mathbb{Q}} \ \text{ for some } R \in \mathcal{P}^e \text{ and } Z_s = 1 \ \forall s \leq t\} \\ &= \{Z \ : \ Z = \frac{dR}{d\mathbb{Q}} \ \text{ for some } R \in \mathcal{P}^e \text{ and } \mathbb{Q} = R \text{ on } \mathcal{F}_t\} \\ &= \{Z \ : \ Z_u = \frac{Z^R_{t \vee u}}{Z^R_t} \text{ where } Z^R_u = \mathbb{E}_u\left[\frac{dR}{d\mathbb{Q}}\right], \ R \in \mathcal{P}^e\}. \end{aligned}$$

The final equality was proved in detail in discrete time. Same proof would also work in continuous time (exercise!)

Using the above, we rewrite the essential supremum as follows,

$$\begin{aligned} U_t : &= \operatorname*{ess\,sup}_{R \in \mathcal{P}^e} \ \mathbb{E}^R_t(H) = \operatorname*{ess\,sup}_{R \in \mathcal{P}^e} \ \mathbb{E}^{\mathbb{Q}}_t[H \frac{Z^R_T}{Z^R_t}] \\ &= \operatorname*{ess\,sup}_{Z \in \mathcal{Z}_t} \ \mathbb{E}^Q_t[HZ_T]. \end{aligned}$$

It is straightforward to check that the family

$$\{\mathbb{E}^{\mathbb{Q}}_t[HZ_T] \ : \ Z \in \mathcal{Z}_t\}$$

is upward-directed. Hence, there exists $\{Z^n\}_{n=1}^\infty \subset \mathcal{Z}_t$ so that

$$\mathbb{E}^{\mathbb{Q}}_t[HZ^n_T] \ \uparrow \ U_t.$$

This implies that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}_s[U_t] &= \lim_{n \uparrow \infty} \mathbb{E}^{\mathbb{Q}}_s[\mathbb{E}^{\mathbb{Q}}_t[HZ^n_T]] \quad \text{ for } s \leq t \\ &= \lim_{n \uparrow \infty} \mathbb{E}^{\mathbb{Q}}_s[HZ^n_T] \\ &\leq \operatorname*{ess\,sup}_{Z \in \mathcal{Z}_s} \ \mathbb{E}^{\mathbb{Q}}_s[HZ_T] \quad \text{ since } Z^n \in \mathcal{Z}_t \subset \mathcal{Z}_s \\ &= U_s. \end{aligned}$$

■

Now we may use the uniform Doob-Meyer Decomposition. Hence, there are $\vartheta \in \Theta_{adm}$, C adapted, non-decreasing, $C_0 = 0$ such that

$$U = U_0 + \int \vartheta dS - C.$$

Since $C_T \geq C_0 = 0$ and $U_T = H$,

$$U_0 + \int^T \vartheta dS_u = U_T + C_T = H + C_T \geq H.$$

Hence, the initial condition U_0 is super-replicating. So we have proved that

$$U_0 = \sup_{\mathbb{Q}\in\mathcal{P}^e} \mathbb{E}^{\mathbb{Q}}[H] \geq \Pi_{seller}(H).$$

Since we have already proved the opposite inequality, we conclude that they are equal. We state this in the following theorem

Theorem 6.3 *(ElKaroui & Quenez) Suppose that $\mathcal{F}_0$ is trivial and the set $\mathcal{P}^e$ is non-empty. Then, for any $H \in \mathcal{L}^0)_+(\mathcal{F}_T)$,*

$$\Pi_{seller}(H) = \sup_{\mathbb{Q}\in\mathcal{P}^e} \mathbb{E}^{\mathbb{Q}}[H].$$

Moreover, if $\Pi_{seller}(H)$ is finite, then the above supremum is achieved and there exists a superhedging admissible strategy.

6.2 Buyer's price

$$\begin{aligned}\Pi_{buy}(H) &:= -\Pi_{sell}(-H)\\ &\Rightarrow = -\sup_{\mathbb{Q}\in\mathcal{P}^e} \mathbb{E}^{\mathbb{Q}}(-H) = \inf_{\mathbb{Q}\in\mathcal{P}^e} \mathbb{E}^{\mathbb{Q}}(H).\end{aligned}$$

This implies that we have a *no-arbitrage price interval*,

$$[\Pi_{buy}(H), \Pi_{sell}(H)] = [\inf \mathbb{E}^{\mathbb{Q}}(H), \sup \mathbb{E}^{\mathbb{Q}}(H)].$$

The above discussion also implies that the theorem that we have proved in finite discrete time also generalizes to continuous time. We first recall the definition.

Definition 6.4 We call $H \in \mathcal{L}^0_+(\mathcal{F}_T)$ attainable if

1. $H = H_0 + G_T(\vartheta)$ for some $H_0 \in R$, $\vartheta \in \Theta_{adm}$;
2. There exists $\mathbb{Q}^* \in \mathcal{P}^e$ such that $G_T(\vartheta)$ is a $\mathbb{Q}^*$ -martingale.

Then, we have the following continuous time extension.

Lemma 6.5 *$H \in \mathcal{L}^0_+$ is attainable if and only if*

$$\sup_{\mathbb{Q}\in\mathcal{P}^e} \mathbb{E}^{\mathbb{Q}}(H) = \mathbb{E}^{\mathbb{Q}^*}(H) < \infty \quad \textit{for some} \quad \mathbb{Q}^* \in \mathcal{P}^e.$$

Proof. $\Rightarrow$: $\mathbb{E}^{\mathbb{Q}^*}(H) = H_0$ and $\mathbb{E}^{\mathbb{Q}}(H) \leq H_0$, for every $\mathbb{Q} \in \mathcal{P}^e$.

$\Leftarrow$: Let U_t be as before (i.e. $=$ ess sup ...). Then, by uniform decomposition, there exists (V_0, φ) so that

$$U_t = V_0 + \int \vartheta dS_u - C.$$

Hence,

$$\mathbb{E}^{\mathbb{Q}}(H) = V_0 + \mathbb{E}^{\mathbb{Q}}\left[\int^T \vartheta dS - C_T\right] \quad \forall \mathbb{Q} \in \mathcal{P}^e.$$

$\blacksquare$

6.3 Portfolio Constraints

Lecture 14, November 7, Tuesday

Chapter 7

American Options.

Lecture 15, November 10, Thursday

As usual $(\Omega, \mathbb{F}, P)$ on $[0,T]$ given, and $\mathcal{P}^e =$ is non-empty.

For a European option we need $\xi \in \mathcal{F}_T$ mbl r.v., e.g. $\xi = (\tilde{S}_T - K)^+$. The holder of an American option, however, can exercise at any stopping time $\tau \in [0,T]$. Hence, we need a payoff process cadlag $U = (U_t)_{0 \leq t \leq T} \geq 0$, e.g. $U_t = (\tilde{S}_t - K)^+$.

Let $\mathscr{S}_{t,T}$ be the set of all stopping times in $[t,T]$. Intuitively the selling price at time t is given by

$$\begin{aligned} \overline{V}_t &:= \operatorname*{ess\,sup}_{\substack{\mathbb{Q} \in \mathcal{P}^e \\ \tau \in \mathscr{S}_{[0,T]}}} \chi_{\{\tau \geq t\}} \mathbb{E}^{\mathbb{Q}}[U_\tau | \mathcal{F}_t] \\ &= \operatorname*{ess\,sup}_{\substack{\mathbb{Q} \in \mathcal{P}^e \\ \tau \in \mathscr{S}_{[t,T]}}} \mathbb{E}^{\mathbb{Q}}[U_t | \mathcal{F}_t]. \end{aligned}$$

Proposition 7.1 *Assume $\mathcal{F}_0$ is trivial and $\overline{V}_0 < \infty$. Then, $\overline{V}$ is a $\mathcal{P}^e$-supermartingale.*

Moreover, $\overline{V}$ is the smallest cadlag processes with this property: if V' is a cadlag process so that V' is a $\mathcal{P}^e$-supermartingale and $V' \geq U$, then $V' \geq \overline{V}$.

Proof. Introduce (fix $\mathbb{Q} \in \mathcal{P}^e$)

$$\mathcal{Z}_t := \{Z \mid Z = \frac{dR}{d\mathbb{Q}} \text{ for some } \; R \in \mathcal{P}^e \text{ and } Z_u = 1 \text{ on } u \in [0,t]\}$$

Then, as before

$$\overline{V}_t = \operatorname*{ess\,sup}_{\substack{Z \in \mathcal{Z}_t \\ \tau \in \mathscr{S}_{[t,T]}}} \mathbb{E}^{\mathbb{Q}}[U_\tau Z_\tau | \mathcal{F}_t].$$

Again using upward-directedness, we find a sequence (Z^n, τ^n). To prove upward-directedness, for any $(Z^i, \tau^i), i = 1,2$ set

$$A := \{\mathbb{E}_t^{\mathbb{Q}}(Z^1_{\tau_1} U^1_{\tau_1}) \geq \mathbb{E}_t^{\mathbb{Q}}(Z^2_{\tau_2} U^2_{\tau_2})\},$$

and

$$\tau := \tau^1 \chi_A + \tau^2 \chi_{A^c} \in \mathscr{S}_{[t,T]} \quad \text{and} \quad Z := Z^1 \chi_A + Z^2 \chi_{A^c} \in \mathcal{Z}_t.$$

Then,

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}[Z_{\tau_1}^1 U_{\tau_1}^1] \vee \mathbb{E}_t^{\mathbb{Q}}(Z_{\tau_2}^2 U_{\tau_2}^2) &= \mathbb{E}_t^{\mathbb{Q}}\left[(Z_{\tau_1}^1 U_{\tau_1}^1)\chi_A\right] + \mathbb{E}_t^{\mathbb{Q}}\left[(Z_{\tau_2}^2 U_{\tau_2}^2)\chi_{A^c}\right] \\ &= \mathbb{E}_t^{\mathbb{Q}}[U_\tau Z_\tau].\end{aligned}$$

Hence, there is a sequence Z_n so that

$$\overline{V}_t = \lim_{n\uparrow\infty} \mathbb{E}_t^{\mathbb{Q}}[U_{\tau_n}^n Z_{\tau_n}^n] \quad \text{(monotone limit)}.$$

We use this to compute,

$$\begin{aligned}\mathbb{E}_s^{\mathbb{Q}}[\overline{V}_t] &= \lim \mathbb{E}_s^{\mathbb{Q}}[\mathbb{E}_t^{\mathbb{Q}}[U_{\tau_n}^n Z_{\tau_n}^n]] \\ &= \lim \mathbb{E}_s^{\mathbb{Q}}[U_{\tau_n}^n Z_{\tau_n}^n] \\ &\leq \overline{V}_s.\end{aligned}$$

Note that

$$\mathbb{E}_0^{\mathbb{Q}}[\overline{V}_t] = \mathbb{E}^{\mathbb{Q}}[\overline{V}_t] \leq \overline{V}_0 < \infty.$$

We have proved that $\overline{V}$ is a $\mathcal{P}^e$-supermartingale.

Minimality of $\overline{V}$. Let V' be a $\mathcal{P}^e$-supermartingale. The, for any $\mathbb{Q} \in \mathcal{P}^e, t$ and $\tau \in \mathscr{S}_{[t,T]}$,

$$V_t' \geq \mathbb{E}_t^{\mathbb{Q}}[U_t] \quad \Rightarrow \quad V_t' \geq \operatorname*{ess\,sup}_{\mathbb{Q},\ \tau} \mathbb{E}_t^{\mathbb{Q}}[U_t] = \overline{V}_t.$$

In the second step , we used the cadlag property. Indeed, $V_t' \geq U_t$ for every t and it is cadlag. Then, we also have $V_\tau' \geq U_\tau$ for every stopping time τ. This is an important point!

Regularity of $\overline{V}$. Note that $\mathcal{F}_t$ is right continuous and complete and $\mathbb{Q} \in \mathcal{P}^e$'s are equivalent to each other. Therefore, a cadlag version can be constructed. Idea here is by backward supermartingale techniques

$$\lim_{t_n \downarrow t} E_{t_n}^Q[\xi],$$

exists. We declare this as the cadlag version. Of course one needs to prove that this limit is a version of the original process. ■

In what follows, we always work with this cadlag version. We know that then $\overline{V}$ is the smallest in the class

1. $\overline{V}$ is cadlag,
2. $\overline{V}$is $\mathcal{P}^e$-supermartingale,
3. $\overline{V} \geq U$.

Let $\Pi_{seller}(U)$ be the superreplication cost of (U_t), i.e.,

$$\Pi_{sel}(U) := \inf\{V_0 \in \mathbb{R} \ : \ \exists\, \vartheta \in \Theta_{adm} \text{ s.t. } V_0 + G(\vartheta) \geq U\}.$$

Notice that since U is cadlag (by assumption) and $G(\vartheta)$ is cadlag (by construction) we conclude that

$$V_0 + G_\tau(\vartheta) \geq U_\tau \quad a.s.,$$

for any $\tau \in \mathscr{S}_{[0,T]}$.

Theorem 7.2 *Assume $\mathcal{P}^e \neq \emptyset$, $\mathcal{F}_0$ is trivial and $\overline{V}_0 < \infty$. Then,*

$$\Pi_{sel}(U) = \sup_{\mathbb{Q}\in\mathcal{P}^e, \tau\in\mathscr{S}_{[0,T]}} \mathbb{E}^{\mathbb{Q}}[U_\tau] = \overline{V}_0.$$

Moreover, there exists $\vartheta \in \Theta_{adm}$ so that

$$\overline{V}_0 + G(\vartheta) \geq U.$$

Proof. Since $\overline{V}$ is a $\mathcal{P}$-supermartingale by optional decomposition theorem, there are $\vartheta \in \Theta_{adm}$ and an adapted, non-decreasing C with $C_0 = 0$ so that

$$\overline{V} = \overline{V}_0 + G(\vartheta) - C.$$

By definition $\overline{V} \geq U$, hence

$$\overline{V}_0 + G(\vartheta) = \overline{V} + C \geq U + C \geq U$$

This proves the second statement. To prove the first statement, suppose that for some $\vartheta \in \Theta_{adm}$ and $x \in \mathbb{R}$,

$$x + G(\vartheta) \geq U.$$

Then, for any $\mathbb{Q} \in \mathcal{P}^e$, $G(\vartheta)$ is a $\mathbb{Q}$-local martingale and for any $\tau \in \mathscr{S}_{[0,T]}$,

$$x \geq \mathbb{E}^{\mathbb{Q}}(x + G_\tau(\vartheta)) \geq \mathbb{E}^{\mathbb{Q}}(U_\tau).$$

Since this holds for every $\mathbb{Q} \in \mathcal{P}^e$, $\tau \in \mathscr{S}_{[0,T]}$ and for every x from which we can super-replicate,

$$\Pi_{sel}(U) \geq \sup_{\substack{\mathbb{Q}\in\mathcal{P}^e \\ \tau\in\mathscr{S}_{[0,T]}}} \mathbb{E}^{\mathbb{Q}}(U_\tau) = \overline{V}_0.$$

Since the second statement proves the opposite inequality this completes the proof. ∎

Interpretation. Set $x = \overline{V}_0$ and $\vartheta \in \Theta_{adm}$ be as in the second part of the theorem. Hence,

$$x + G(\vartheta) = V^{x,\vartheta} \geq U,$$

So the seller is always safe and may even profit if option is exercised at an non-optimal τ with

$$x + G_\tau(\vartheta) - U_\tau > 0.$$

Exercise time. The holder should exercise at a time

$$\overline{V}_\tau = U_\tau.$$

Otherwise he would receive U_τ \$'s for something worth $\overline{V}_\tau$ \$'s. Then, we may define

$$\tau^* := \inf\{\ t \in [0,T]\ :\ \overline{V}_t = U_t\},$$

7.1 Markov structure

Consider the standard Black & Scholes case,

$$d\tilde{S}_t = \tilde{S}_t(\mu dt + \sigma dW_t), \quad d\tilde{B}_t = r\tilde{B}_t dt, \quad U_t = \varphi(\tilde{S}_t).$$

The option value is given by

$$\overline{V}_t := \operatorname*{ess\,sup}_{\tau \in \mathscr{S}_{[t,T]}} \mathbb{E}^{\mathbb{Q}^*} \varphi(\tilde{S}_\tau).$$

We know that under the risk neutral measure $\mathbb{Q}^*$

$$d\tilde{S}_t = \tilde{S}_t(rdt + \sigma dW_t^*),$$

where

$$W_t^* = W_t + \frac{\mu - r}{\sigma} t, \quad \mathbb{Q}^*\text{-Brownian motion.}$$

Then,

$$\overline{V}_t = v(t, \tilde{S}_t),$$

where

$$\begin{aligned} V(t,y) &:= \sup_{\tau \in \mathscr{S}_{[t,T]}} \mathbb{E}^{\mathbb{Q}^*} \left[\varphi(Y_\tau) e^{-r(\tau - t)} \mid Y_t = y \right] \\ dY_t &= Y_t(rdt + \sigma dW_t^*). \end{aligned} \tag{7.1.1}$$

Lemma 7.3 *V is the unique solution of*

1. $v \in W^{1,2,\infty}_{loc}((0,T) \times (0,\infty)), \quad 0 \le v(t,y) \le y.$
2. $v(t,0) = 0, \quad \forall\ t \in [0,T]$, *(true but not really needed).*
3. $\min\{-v_t + rv - ryv_y - \frac{1}{2}\sigma^2 y^2 v_{yy}\ ;\ v(t,y) - \varphi(y)\} = 0, \quad \forall t \in (0,T), y > 0.$
4. $v(T,y) = \ varphi(y), \quad \forall\ y \ge 0.$

Note that $W^{1,2,\infty}(Q) = \{v, \frac{\partial v}{\partial t}, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial y^2} \ \in L^\infty(Q)\}$.
Proof. Set

$$J(t,y,\tau) := \mathbb{E}^{\mathbb{Q}^*}(\ varphi(Y_t)\chi_{\{t \le \tau\}})$$

Let v be a solution as above. Then, Ito's rule can be applied locally. Fix (t,y) and consider an open set ϑ_n such that

$$(t,y) \ni \vartheta_n \subset \vartheta_{n+1} \subset (0,T) \times (0,\infty)$$

and

$$\bigcup_n \vartheta_n = (0,T) \times (0,\infty)$$

Let $Y^{y,t}$ be the solution of (7.1.1) with initial data $Y_t^{y,t} = y$. Let $\hat{\tau}_n$ be the exit time of $Y^{y,t}$ f rom ϑ_n. Then, by the Ito formula, for any $\tau \in \mathscr{S}_{[t,T]}$,

$$e^{-r(\tau_n - t)} v(\tau_n, Y_{\tau_n}) = v(t,y) + \int_t^{\tau_n} e^{-r(u-t)} [v_t - rv + \mathcal{L}v] du + \int_t^{\tau_n} (\ldots) d\tilde{W}_t,$$

where $\tau_n := \hat{\tau}_n \wedge \tau$. Hence,

$$v(t,y) = \mathbb{E}^{\mathbb{Q}^*}[e^{-r(\tau_n - t)} v(\tau_n, Y_{\tau_n}) - \int_t^{\tau_n} e^{-r(u-t)} [v_t - rv + \mathcal{L}v] du].$$

By the partial differential equation,

$$(v_t - rv + \mathcal{L}v)(t', y') \le 0, \quad v \ge (K - y'),$$

and

$$\begin{aligned} v(t,y) &\ge \mathbb{E}^{\mathbb{Q}^*}[e^{-r(\tau_n - t)} v(\tau_n, Y_{\tau_n})] \\ &\ge \mathbb{E}^{\mathbb{Q}^*}[e^{-r(\tau \wedge \hat{\tau}_n - t)} \varphi(Y^{y,t}_{\tau \wedge \hat{\tau}_n})], \quad \forall n \text{ and } \tau \in \mathscr{S}_{[t,T]}. \end{aligned}$$

We may let $n \uparrow \infty$ to conclude that any solution v satisfies

$$v \ge V = \text{ value function.}$$

The other inequality is proved by choosing τ appropriately. Set

$$\mathscr{C} := \{ (t', y') \ : \ v(t', y') > \varphi(y) \ \}.$$

It is an open set.
(1) If $(t,y) \notin \mathscr{C}$ choose $\tau = t$. Then,

$$v(t,y) = \varphi(y) \le V(t,y).$$

(2) If $(t,y) \in \mathscr{C}$, then choose τ to be the exit time from $\mathscr{C}$. Then, for $t' \in [t, \tau]$,

$$(v_t - rv + \mathcal{L}v)(t', Y_{t'}) = 0 \quad a.s.,$$

and

$$v(t,y) = \mathbb{E}^{\mathbb{Q}^*}[e^{-r(\tau_n - t)} v(\tau_n, Y_{\tau_n})].$$

Now

$$\lim v(\tau_n, Y_{\tau_n}) = v(\tau, Y_{\tau_n}) = \varphi(Y_{\tau_n})$$

and

$$0 \le v(\tau_n, Y_{\tau_n}) \le Y_{\tau_n}$$

so by dominated convergence,

$$v(t,y) = \mathbb{E}^{\mathbb{Q}^*}(e^{-r(\tau - t)} \varphi(Y_\tau)) \le \overline{V}(t,y).$$

∎

Lecture 16, November 14, Tuesday

Existence: In the literature this is known as

(i) obstacle problem:

$$\min\{-\Delta u,\ u-\varphi\}=0, \quad x\in\vartheta\subset\mathbb{R}^n.$$

(ii) Stefan problem:

$$\min\{u_t-u_{xx},\ u\}=0.$$

This is a model for solid to liquid phase transition. The solution u is the temperature and the freezing temperature is normalized to zero,

Regularity:
$W^{1,2,\infty}$ is the best regularity. Indeed, consider a one-dimensional problem, with $f(x)=1-x^2$ and

$$\min\{-u_{xx},\ u-f\}=0 \quad 0<x<2, \quad u(2)=0.$$

Let $x_0\in(0,2)$ be such that

$$u_{xx} \text{ on } (x_0,2), \quad u(x_0)=f(x_0), \quad u(2)=0.$$

Then

$$u(x)=f(x_0)-\frac{f(x_0)}{2-x_0}(x-x_0)=f(x_0)\left[1-\frac{x-x_0}{2-x_0}\right]=f(x_0)\frac{2-x}{2-x_0}$$

The "minimality" of u implies the *smooth fit.* Namely at x_0, $u'(x_0)=f'(x_0)$. This implies that

$$\begin{aligned}
\Rightarrow \quad -f(x_0)\frac{1}{2-x_0} &= f'(x_0)=-2x_0\\
\Rightarrow \quad (1-x_0^2) &= -2x_0(2-x_0)=-4x_0+x_0^2\\
\Rightarrow \quad 3x_0^2-4x_0-1 &= 0\\
\Rightarrow \quad x_0 &= (2\pm\sqrt{(7)})/3<2.
\end{aligned}$$

We can check that u solves the equation but

$$u''(x_0^+)=0, \quad u''(x_0^-)=f''(x_0)=-2.$$

7.2 Example: American call option.

Suppose $\mathcal{P}=\{\mathbb{Q}^*\}$ and $S=\tilde{S}/\tilde{B}$ is a true $\mathbb{Q}^*$-martingale. Consider $\tilde{U}_t:=(\tilde{S}_t-K)^+$, $0\le t\le T$. Then, we claim that if $\tilde{B}_t$ is increasing, then

$$\tilde{V}_t=\tilde{B}_t\mathbb{E}^{\mathbb{Q}^*}\left[\frac{(\tilde{S}_T-K)^+}{\tilde{B}_T}|\mathcal{F}_t\right]=\tilde{B}_t\mathbb{E}^{\mathbb{Q}^*}\left[\frac{\tilde{U}_T}{\tilde{B}_T}|\mathcal{F}_t\right],$$

which means that the American call is not more valuable than the European call.
Proof. This essentially uses that S is a $\mathbb{Q}^*$-martingale and $(x-K)^+$ is convex, and that a convex function of a martingale is a submartingale. More precisely, assume that $\tilde{B}$ is increasing. Then

$$\frac{\tilde{U}_\tau}{\tilde{B}_\tau} = \left(S_\tau - \frac{K}{\tilde{B}_\tau}\right)^+ \geq \left(S_\tau - \frac{K}{\tilde{B}_t}\right)^+ \quad \text{for } \tau \in \mathscr{S}_{t,T}.$$

By the Jensen's inequality,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^*}\left[\frac{\tilde{U}_\tau}{\tilde{B}_\tau}|\mathcal{F}_t\right] &\geq \left(\mathbb{E}^{\mathbb{Q}^*}\left[S_\tau - \frac{K}{\tilde{B}_t}|\mathcal{F}_t\right]\right)^+ \\ &= \left(S_t - \frac{K}{\tilde{B}_t}\right)^+ = \frac{\tilde{U}_t}{\tilde{B}_t} \end{aligned}$$

So $\tilde{U}/\tilde{B}$ is a $\mathbb{Q}^*$-submartingale. Hence,

$$\mathbb{E}^{\mathbb{Q}^*}\left[\frac{\tilde{U}_T}{\tilde{B}_T}|\mathcal{F}_\tau\right] \geq \frac{\tilde{U}_\tau}{\tilde{B}_\tau} \quad \text{for } \tau \in \mathscr{S}_{t,T},$$

and so the esssup over τ is obviously attained for $\tau \equiv T$. (This reflects the fact that one should never stop a submartingale, since it grows on average.) ■

Put options. Since $(K-x)^+$ is also convex, one might expect an analogous for the American put option. If $\tilde{B} \equiv 1$, this is correct. But if $\tilde{B}$ is really increasing (i.e., interest rates are positive), then an American put option is typically worth strictly more than the corresponding European put option.
Exercise: Show that in binomial model with $r > 0$, $\tilde{V}_0^{am} > \tilde{V}_0^{eur}$ for all sufficiently large K.

Lecture 17, November 17, Thursday

No Class.

Chapter 8

Jump Markov processes.

Lectures 18-19, November 21, 24

Chapter 9

Merton Problem

This is a classical problem in optimal investment and consumption. We assume that the financial market consists of one risk and one non-risky asset. The returns are given by,

$$
\begin{aligned}
dS_t &= S_t[rdt + \sigma(\lambda dt + dW_t)] \\
dB_t &= B_t r dt.
\end{aligned}
$$

If this investor (or the representative agent) chooses to invest π_t shares of her wealth in the stock and consumes at a rate of C_t, then ther welath, or equivalently tne marked-to-market value of her portfolio, evolves according to

$$
dY_t = Y_t[rdt + \pi_t\sigma(\lambda dt + dW_t)] - C_t dt.
$$

We call $(\pi, C) \in \mathcal{A}_{adm}$ if it is adapted to the Brownian filtration and satisfies

$$
\begin{aligned}
&1)\ \textstyle\int_0^t \pi_u^2 du < \infty \quad \int_0^t C_u du < \infty, \quad \mathbb{P}-a.s.,\ \forall t > 0, \\
&2)\ Y_t^{\pi,C} \geq 0, \quad \mathbb{P}-a.s., \forall t \geq 0.
\end{aligned}
$$

Then, her goal is to maximize her utility from consumption given by

$$
J(y, \pi, C) := \mathbb{E} \int_0^\infty e^{-\beta t} U(C_t) dt, \quad Y_0 = y,
$$

where

$$
U : [0, \infty) \to \mathbb{R}
$$

is a utility function, i.e. a non-decreasing and concave function.

Remark 9.1 It is better to define

$$
c_t := \frac{C_t}{Y_t}
$$

Then $(\pi, c) \in \mathcal{A}_{adm}$ at any $y > 0$ implies that $(\pi, c) \in \mathcal{A}_{adm}$ for any other initial point. Indeed

$$
Y^{y,\pi,c} = \frac{y}{y'} Y^{y',\pi,c}
$$

for any $y, y' > 0$. So $Y^{y,\pi,c} > 0$ iff $Y^{y',\pi,c} > 0$ and

$$
\textstyle\int_0^t c_u Y_u^y du = \frac{y}{y'} \int_0^t c_u Y_u^{y'} du.
$$

Set

$$v(y) := \inf_{(\pi,c)\in\mathcal{A}_{adm}} J(y,\pi,c)$$

An important class of utility functions are the so-called CRRA class,

$$U(c) := \frac{1}{1-\gamma}c^{1-\gamma} \quad \text{for } c>0, \gamma \neq 0,\ \gamma<1. \quad (\text{or } = \ln c)$$

Then, it is straigtforward to show that

$$J(\alpha y,\pi,c) = \alpha^{1-\gamma}J(y,\pi,c) \quad \forall(\pi,c)\in\mathcal{A}_{adm} \text{ and } \alpha>0.$$

Hence,

$$v(\alpha y) = \alpha^{1-\gamma}v(y), \Rightarrow\ v(y) = y^{1-\gamma}v(1).$$

9.1 Dynamic Programming

In this section, we formally derive a partial differential equation satisfied by the value function. This derivation holds for all utility functions. However in the case of a CRRA utility, then one can solve this differential equation explicitly.

We start with the dynamic programming principle. For any stopping time $\theta>0$,

$$v(y) = \inf \mathbb{E}\left[\int_0^\theta e^{-\beta t}U(c_tY_t)dt + e^{-\beta\theta}U(Y_\theta)\right].$$

We now obtain, by the Ito rule,

$$e^{-\beta\theta}v(Y_\theta) = v(y) + \int_0^\theta e^{-\beta t}[-\beta v + Y_t(rv' + \pi_t\sigma\lambda v') + \frac{\sigma^2}{2}\pi_t^2Y_t^2v'' - c_tY_tv']dt + \ ''\text{martingale}''$$

Ignoring the technical details, we take $\theta = h \ll 1$, substitute the Ito calculation into the dynamic programming principle and take the expectation. The result is, and

$$0 = \sup\ \mathbb{E}\frac{1}{h}\int_0^h e^{-\beta t}[-\beta v + Y_tv'(r+\pi_t\sigma\lambda - c_t) + Y_t^2\frac{\sigma^2}{2}\pi_t^2v'' + U(c_tY_t)]dt.$$

Now, formal passage to the limit as h tends to zero yields,

$$\beta v + \inf_{\pi\in\mathbb{R}^1}[-(\pi\sigma)\lambda yv' - \frac{\gamma^2}{2}(\pi\sigma)^2y^2v''] + \sup_{c\geq 0}[(cy)v' - U(cy)] - rv'y = 0.$$

The above is the *dynamic programming equation* for this optimization problem. It holds for every utility function. However, in the case of a power utility, the value function has the simple form,

$$v(y) = y^{1-\gamma}v(1) =: \frac{A}{1-\gamma}y^{1-\gamma}.$$

Then, using the dynamic programming equation, we can solve for the constant A explicetly. Indeed,

$$y^{1-\gamma}\left\{A\left[\frac{\beta}{(1-\gamma)} + \inf_{\tilde{\pi}}(-\tilde{\pi}\lambda + \frac{\gamma}{2}\tilde{\pi}^2 - r\right] + \inf_{\hat{c}>0}\left(A\hat{c} - \frac{1}{1-\gamma}(\hat{c})^{1-\gamma}\right)\right\} = 0.$$

We directly calculate that

$$\tilde{\pi}^* = \pi^* \sigma = \frac{\lambda}{\gamma} \Leftrightarrow \pi^* = \frac{\lambda}{\sigma\gamma}, \quad \text{and} \quad \text{min value} = -\frac{\lambda^2}{2\gamma},$$
$$c^* = A^{-\frac{1}{\gamma}}, \quad \text{and} \quad \text{min value} = -\frac{\gamma}{1-\gamma} A^{\frac{\gamma-1}{\gamma}} = -\frac{\gamma}{1-\gamma} A^{1-\frac{1}{\gamma}}.$$

Hence,

$$A\left[\frac{\beta}{(1-\gamma)} - r - \frac{\lambda^2}{2\gamma}\right] - \frac{\gamma}{1-\gamma} A^{1-\frac{1}{\gamma}} = 0.$$
$$\Rightarrow \quad A = \left(\frac{(1-\gamma)}{\gamma}\left[\frac{\beta}{(1-\gamma)} - r - \frac{\lambda^2}{2\gamma}\right]\right)^{-\gamma}.$$

In summary,

$$\pi^* = \frac{\lambda}{\sigma\gamma}, \quad c^* = A^{-\frac{1}{\gamma}} = \frac{1-\gamma}{\gamma}\left(\frac{\beta}{(1-\gamma)} - r - \frac{\lambda^2}{2\gamma}\right).$$

Theorem 9.2 *The value function and the optimal portfolio and the consumption rates are given as above.*

Proof. Straightforward use of Ito's rule and the dynamic programming equation. We outline the proof quickly.

First, we note that the value function is finite if and only if

$$\beta > r(1-\gamma) + \frac{\lambda^2}{2}\frac{(1-\gamma)}{\gamma}.$$

The dynamic programming equation implies that for any consumption process c_t and any π,

$$\frac{\beta}{\gamma} + \frac{1}{2}\pi^2\sigma^2(1-\gamma) - \pi\lambda\sigma - r + c_t \geq \frac{1}{\gamma}(c_t)^\gamma \frac{1}{A}.$$

Let Y be the wealth process correspinding to an arbitrary strategy c_t, π_t. then,

$$d(e^{-\beta t}\frac{A}{\gamma}(Y_t)^\gamma) \leq e^{-\beta t}\frac{(c_t Y_t)^\gamma}{\gamma} dt + e^{-\beta t}\frac{A}{\gamma}(Y_t)^\gamma \sigma \pi_t dW_t,$$

or equivalently

$$e^{-\beta t}\frac{A}{\gamma}(Y_t)^\gamma + \int_0^t e^{-\beta u}\frac{(c_u Y_u)^\gamma}{\gamma} du \leq \frac{A}{\gamma}(Y_0)^\gamma + \underbrace{\int_0^t e^{-\beta u}(Y_u)^\gamma \pi_u dW_u}_{:=N_t} \frac{A\sigma}{\gamma}.$$

Observe that N_t is a local martingale, $EN_t \leq 0$. Hence,

$$\lim_{t\to\infty} E\int_0^t e^{-\beta u}\frac{(c_u Y_u)^\gamma}{\gamma} du \leq \frac{A}{\gamma}(Y_0)^\gamma$$
$$\Rightarrow \quad J(Y_0; \pi, c) \leq \frac{A}{\gamma}(Y_0)^\gamma.$$

To prove the optimality of π^* and c^* we observe that all the inequalitiesin the above calculation are euqlaities when we use the cosntant rate c^* and π^*. ∎

9.2 Utility Indifference Price

Definition 9.3 Given $\xi \in L_T$ mbl and $x \in \mathbb{R}^+$, $p \in \mathbb{R}$ is the *marginal utility indifference price* if

$$u(x) \geq EU(x - \lambda p + \int^T \vartheta_u dS_u + \lambda\xi) \quad \forall \vartheta \in \Theta_{adm}, \lambda \in \mathbb{R}.$$

For a fixed $= \lambda$ (say $= 1$), the utility indifference price p_λ of λ shares is given by

$$u(x) = \sup_{\vartheta \in \Theta_{adm}} \mathbb{E}U(x - p_\lambda + \xi + \int^T \vartheta_u du)$$

The idea is that the investor is indifferent to buying it for p and holding it or not.

It is not clear that the marginal price exists.

Lemma 9.4 *Suppose that there exists a unique martingale measure $\mathbb{Q}$, i.e. $\mathcal{M} = \{\mathbb{Q}\}$. Then, for every $\xi \in L^\infty(\mathcal{F}_T)$, the marginal utility indifference price is equal to the classical Black-Schole sprice,*

$$p^{BS} = \mathbb{E}^{\mathbb{Q}}[\xi].$$

Proof.

Under the assumptions we know that there exists $\vartheta^{BS} \in \Theta_{adm}$ so that

$$\xi = p^{BS} \int^T \vartheta_u^{BS} dS_u.$$

Now let $\vartheta \in \Theta_{adm}$ be arbitrary. Then,

$$x - \lambda p^{BS} + \lambda\xi + \int^T \vartheta_u dS_u = x + \int^T (\vartheta_u + \lambda \vartheta_u^{BS}) dS_u.$$

Since $\vartheta + \lambda\vartheta^{BS} \in \Theta_{adm}$, we conclude. ∎

So the interesting stuff is when not complete or when $\vartheta^{BS} \notin \Theta_{adm}$. But this price is x dependent and the price for fix λ is also λ and x dependent.

Suppose that a maximizer ϑ^* exists, i.e.

$$u(x) = \mathbb{E}U(x + \int^T \vartheta_u^* dS_u).$$

By definition of the marginal price p,

$$\mathbb{E}U(x + \int^T \vartheta_u^* dS_u) \geq \underbrace{\mathbb{E}U(x - \lambda p + \lambda\xi + \int^T \vartheta_u^* dS_u)}_{:=H(\lambda)}, \quad \lambda \in \mathbb{R}.$$

Hence $\lambda \to H(\lambda)$ is maximized at the origin:

$$\mathbb{E}[U'(x + \int^T \vartheta_u^* dS_u)(\xi - p)] = 0$$
$$\Rightarrow p = \mathbb{E}^{\mathbb{Q}}[\xi]; \quad \frac{dQ}{dP}|_{\mathcal{F}_t} = \frac{U'(x + \int^T \vartheta_u^* dS_u)}{E[U'(...)]}.$$

Therefore the utility indifference prices chooses an equaivalent martingale measure $\mathbb{Q}$.

Another technical point left out aboveis the following: How do we know that the price exists and unique ?

Set

$$\hat{u}(x,\lambda) := \sup_{\vartheta\in\Theta_{adm}} \mathbb{E}U(x + \int^T \vartheta_u dS_u + \lambda\xi).$$

Price p should satisfy

$$\hat{u}(x-\lambda p,\lambda) \le u(x) = \hat{u}(x,0).$$

Lemma 9.5 *$\hat{u}$ is jointly concave in x and λ.*

Proof. $(x_i,\lambda_i) \to (\bar{x},\bar{\lambda})$ is the mid-point. Choose ϑ^i be an ε-maximizer, and set $\bar{\vartheta}$ be again the mid point. Then,

$$\mathbb{E}U(\bar{x} + \int^T \bar{\vartheta}_u dS_u + \bar{\lambda}\xi) \ge \frac{1}{2}\sum_{i=1}^{2} \mathbb{E}U(x^i + \int^T \vartheta^i_u dS_u + \lambda^i\xi).$$

■

Convex analysis implies that

$$\partial\hat{u}(x,0) = \{(z_1,z_2) \ : \ \hat{u}(x',\lambda') \le \hat{u}(x,0) + z_1(x'-x) + z_2\lambda', \ \forall x',\lambda'\}.$$

We know (by Hahn-Banach) that $\partial\hat{u} \neq \emptyset$. Let $z = (z_1,z_2) \in \partial\hat{u}(x,0)$. Set $p = \frac{z_2}{z_1}$ if $z_1 \neq 0$. Then,

$$\hat{u}(x-\lambda p,\lambda) \le \hat{u}(x,0) - \lambda p z_1 + \lambda z_2 = \hat{u}(x,0) = u(x)$$

Note that since U is strictly increasing, we expect $u_x \neq 0$. But there may be non-uniqueness.
Remarks.

1. Marginal definition is useful in the incomplete case to single out a EMM as the pricing kernel. Note that $\mathbb{Q}$ in above is independent of ξ. This makes the result useful.

2. There are other ways of choosing a EMM
   (a) Minimizing entropy (Schweizer);
   (b) Good deal bounds.

3. In the frictional cases, the marginal utility price is most of the time given by the BS price.

9.3 Optimal wealth process

$$\begin{aligned} dY_t &= Y_t\left(\left[rdt + \frac{\lambda}{\gamma}(\lambda dt + dW_t)\right] - K^* dt\right) \\ &= Y_t\left((r + \frac{\lambda^2}{\gamma} - K^*)dt + \frac{\lambda}{\gamma}dW_t\right), \end{aligned}$$

where

$$K^* = \frac{1-\gamma}{\gamma}\left(\frac{\beta}{1-\gamma} - r - \frac{\lambda^2}{2\gamma}\right).$$

Then,

$$Y_t = y\exp\left(\frac{\lambda}{\gamma}W_t - \frac{1}{2}\frac{\lambda^2}{\gamma^2}t + (r + \frac{\lambda^2}{\gamma} - K^*)t\right).$$

The important quantity is,

$$\begin{aligned}
\alpha &:= -\frac{1}{2}\frac{\lambda^2}{\gamma^2} + r + \frac{\lambda^2}{\gamma} - \frac{\beta}{\gamma} + r\frac{(1-\gamma)}{\gamma} + (1-\gamma)\frac{\lambda^2}{2\gamma^2} \\
&= -\frac{\beta}{\gamma} + \frac{r}{\gamma} + \frac{\lambda^2}{2\gamma^2}(-1 + 2\gamma + (1-\gamma)) \\
&= -\frac{\beta}{\gamma} + \frac{r}{\gamma} + \frac{\lambda^2}{2\gamma^2}(\gamma) = \frac{1}{\gamma}(\frac{\lambda^2}{2} + r - \beta).
\end{aligned}$$

Lemma 9.6 $Y_t^* = y_0 \exp\left(\frac{\lambda}{\gamma}W_t - \frac{1}{\gamma}(\beta - r - \frac{\lambda^2}{2})t\right)$.

Proof.

We directly calculate that the density of the risk neutral measure

$$Z_t = \frac{d\mathbb{Q}}{d\mathbb{P}}$$

solves

$$dZ_t = Z_t\left[-\lambda dW_t\right] \;\Rightarrow\; Z_t = e^{-\lambda W_t - \frac{1}{2}\lambda^2 t}.$$

Notice that

$$\begin{aligned}
e^{-\beta t}U'(c^*Y_t^*) &= \text{const. } e^{-\beta t}(Y_t^*)^{-\gamma} \\
&= \text{const. } \exp(-\lambda W_t - rt - \frac{1}{2}\lambda^2 t) \\
&= \text{const. } e^{rt}Z_t.
\end{aligned}$$

We also directly calculate that

$$e^{-\beta t}U(c^*Y_t^*) = \text{constant}\exp\left(\frac{(1-\gamma)}{\gamma}\lambda W_t - \frac{(1-\gamma)}{\gamma}(\beta - r - \frac{\lambda^2}{2})t - \beta t\right).$$

Hence,

$$\begin{aligned}
\mathbb{E}[e^{-\beta t}U(c^*Y_t^*)] &= \text{const.}\exp\left(\frac{(1-\gamma)^2}{2\gamma^2}\lambda t - \frac{(1-\gamma)}{\gamma}(\beta - r - \frac{\lambda^2}{2})t - \beta t\right) \\
&= \text{const.}\exp\left(-\frac{(1-\gamma)}{\gamma}\left[\frac{\beta}{(1-\gamma)} - r - \frac{\lambda^2}{2\gamma}\right]\right).
\end{aligned}$$

This is integrable as

$$\frac{\beta}{1-\gamma} > r + \frac{\lambda^2}{2\gamma}.$$

■

Lemma 9.7

$$v'(y) = Ay^{-\gamma} = EU'(c^*Y_t^*)e^{-\beta t}e^{rt} \quad \forall t$$

Proof. We have calculated that

$$e^{-\beta t}U'(c^*Y_t^*) = \text{cons. } e^{rt}Z_t.$$

Hence,

$$E(e^{-\beta t}U'(c^*Y_t^*)) = \text{cons. } e^{rt}.$$

We need to check that constant$= y^{-\gamma}A$. Indeed,

$$\text{constant} = y_0^{-\gamma}(c^*)^{-\gamma} = y_0^{-\gamma}(A^{-\frac{1}{\gamma}})^{-\gamma} = y_0^{-\gamma}A.$$

■

Now consider a general utility function U. We would like to extend these two observations that

1. $v'(y) = \mathbb{E}[U'(C_\vartheta^*)e^{-\beta\vartheta}e^{r\vartheta}], \quad \forall$ stopping time $\vartheta > 0$
2. $v'(y)S_0 = \mathbb{E}[U'(C_\vartheta^*)e^{-\beta\vartheta}e^{r\vartheta}S_\vartheta]$.

Hence,

$$\frac{U'(C_t^*)e^{-\beta t}e^{rt}}{v'(y)} = Z_t$$

is the state price density.

We assume that there exists a maximizer, given $y \in (0, \infty)$

$$(\pi^*, c^*) \in \mathcal{A}_{adm}$$

so that

$$J(y, \pi^*, c^*) = \max E \int_0^\infty e^{-\beta t}U(c^*Y_t^*)dt = v(y)$$

where U is a general utility function on $[0, \infty)$.

Lemma 9.8

$$v'(y) = \mathbb{E}e^{-\beta\vartheta}e^{r\vartheta}U'(c^*Y_\vartheta^*)$$

for every ϑ.

Proof. Since $\mathcal{A}_{adm}$ does not depend on the initial condition, for $y^\varepsilon = y + \varepsilon$, we consider the strategies

(i) for $t \in [0, \vartheta]$: want

$$Y_t^\varepsilon = Y_t^* + \varepsilon e^{rt}$$
$$\pi_t^\varepsilon Y_t^\varepsilon = \pi_t^* Y_t^*, \quad C_t^\varepsilon = C_t^* \quad (C = cY),$$

(ii) for $t > \vartheta + h$ want

$$Y_t^\varepsilon = Y_t^*,\ \pi_t^\varepsilon = \pi_t^*,\ C_t^\varepsilon = C_t^*,$$

(iii) for $t \in [\vartheta, \vartheta + h]$,

$$\begin{aligned} C_t^\varepsilon &= C_t^* + \frac{\varepsilon}{h} e^{rt} \\ Y_t^\varepsilon &= Y_t^* + \varepsilon e^{rt}(1 - \frac{t-\vartheta}{h}) \\ \pi_t^\varepsilon Y_t^\varepsilon &= \pi_t^* Y_t^* \end{aligned}$$

For $\varepsilon > 0$, $(\pi^\varepsilon, C^\varepsilon) \in \mathcal{A}_{adm}$ as $Y_t^\varepsilon \geq Y_t$, Then,

$$\begin{aligned} v(y+\varepsilon) &\geq J(y+\varepsilon, c^\varepsilon, \pi^\varepsilon) \\ \Rightarrow \quad \frac{v(y+\varepsilon) - v(y)}{\varepsilon} &\geq \frac{1}{\varepsilon}[J(y+\varepsilon, c^\varepsilon, \pi^\varepsilon) - J(y, c^*, \pi^*)] \\ &= \frac{1}{\varepsilon} \mathbb{E} \int_\vartheta^{\vartheta+h} e^{-\beta t} \left[U(C_t^* + \frac{\varepsilon}{h} e^{rt}) - U(C_t^*) \right] dt \end{aligned}$$

$$\lim_{\varepsilon \downarrow 0} \frac{v(y+\varepsilon) - v(y)}{\varepsilon} \geq \lim_{h \downarrow 0} \lim_{\varepsilon \downarrow 0} (\ldots) = \mathbb{E} e^{-\beta\vartheta} e^{r\vartheta} U'(c^* Y_\vartheta^*).$$

For $\varepsilon < 0$, we define

$$\vartheta^\varepsilon := \vartheta \wedge \tau^\varepsilon$$

where τ^ε is the time $Y_t^* + \varepsilon e^{rt} = 0$. We know that $\vartheta^\varepsilon \uparrow \vartheta$ and the above calculation proves the result. ■

On $[0, T]$ define an equivalent measure $\mathbb{Q}^T$ by

$$\frac{d\mathbb{Q}^T}{d\mathbb{P}} |_{\mathcal{F}_T} = e^{-(\beta - r)T} U'(c_T^* Y_T^*), \quad \hat{S}_t = e^{-rt} S_t.$$

Lemma 9.9 *On $[0, T]$, $(\hat{S}_t)_{t \in [0,T]}$ is a $\mathbb{Q}^T$-martingale.*

Proof. We only show that

$$S_0 = e^{-rt} \mathbb{E}^{\mathbb{Q}}[S_t].$$

Some calculation with initial endowment of $y + \varepsilon S_0$. We simply buy ε shares of the stock in addition to our optimal portfolio starting from y. Then assume εS_ϑ in the time interval $[\vartheta, \vartheta + h]$ as above. Same calculation yield,

$$\begin{aligned} v'(y) S_0 &= \mathbb{E} U'(c_\vartheta^* Y_\vartheta^*) S_\vartheta e^{-\beta\vartheta} \\ \Rightarrow \quad S_0 &= \frac{\mathbb{E}\left[U'(c_\vartheta^* Y_\vartheta^*) e^{-(\beta-r)\vartheta} e^{-r\vartheta} S_\vartheta \right]}{v'(y)} \\ &= \mathbb{E}^{\mathbb{Q}}(e^{-r\vartheta} S_\vartheta). \end{aligned}$$

■

Above calculations use

1. existence of an optimal control;
2. $dY_t = (Y_t - Z_t S_t) r dt + Z_t dS_t$.

On the other hand, the fact that r is constant not important and the form of dS is not important either.
Conclusion. In general with $\mathbb{E}\int_0^\infty U_t dt$ and $dB = r_t dB_t$

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = U_t'(c_t^* Y_t^*)\frac{1}{v'(y)B_t}$$

is a risk neutral measure.

9.4 Utility Maximization-Dual Approach

1. We look at it in incomplete market with no friction.
2. We look at the simpler problem with no consumption.

We always assume that

$$\mathcal{M} = \text{set of all equivalent local mart. measures} \neq \emptyset.$$

The probelm is to maximize

$$\mathbb{E}\mathrm{U}(x + \int^T \vartheta_u dS_u)$$

over all $\vartheta \in \Theta_{adm} \Leftrightarrow$ integrable, predictable, satisfying

$$\int \vartheta_u dS_u \geq -C.$$

We assume that the utility function

$$U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$$

(a) increasing,

(b) concave, strict on $\{U :> -\infty\}$,

(c) differentiable on $\{U :> -\infty\}$,

$$U'(\infty) = 0.$$

Two cases are different.

i . $U = -\infty$ on $(-\infty, 0)$ and $\mathrm{U}'(0) = \infty$

$$\text{e.g.}\quad U(x) = \frac{x^{1-\gamma}}{1-\gamma} \;\Rightarrow U' = x^{-\gamma}.$$

ii . $U'(-\infty) = \infty$ and $U > -\infty$ everywhere.

$$\text{e.g.}\quad U(x) = 1 - e^{-\lambda x}.$$

9.4.1 Arrow-Debrue case: Finite probability and complete

the setup is

$$(S_t)_{t=0,...,T},\ \Omega = \{\omega_1, ..., \omega_N\}, \qquad \mathcal{F}_0 = \{\emptyset, \Omega\},\ \mathcal{F}_T = 2^{\Omega}.$$

We assume that

$$\exists \mathbb{Q} : \mathbb{Q}(\omega_n) > 0,$$

and S is a $\mathbb{Q}$-martinagale.

First considet the complete case $\mathcal{M}^e = \{\mathbb{Q}\}$. Then,

$$\begin{aligned}
&\text{maximize } \ \mathbb{E}\mathrm{U}(X_T) = \sum_{n=1}^{N} p_n U(\xi_n) = u(x) \\
&\text{subject to } \xi_n = x + (\textstyle\int^T H dS)(\omega_n) \\
&\quad\Leftrightarrow \mathbb{E}^{\mathbb{Q}} \xi_n = \sum q_n \xi_n = x,
\end{aligned}$$

and $p_n = P(\omega_n)$. So set

$$C(x) := \{\xi \in L^0(\mathcal{F}_T) :\ \mathbb{E}^{\mathbb{Q}}\xi \leq x\}.$$

Our problem is

$$\left.\begin{aligned}
&\text{maximize } \sum_{n=1}^{N} p_n U(\xi_n) \\
&\text{such that } \sum q_n \xi_n \leq x.
\end{aligned}\right\} \qquad \text{(P)}$$

Since $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$ we use a Lagrange multiplier and define

$$\begin{aligned}
L(\xi_1, ..., \xi_n, y) &:= \sum p_n U(\xi_n) - y(\sum q_n \xi_n - x) \\
&= \sum p_n [U(\xi_n) - y \frac{q_n}{p_n} \xi_n] + yx,\ y \in \mathbb{R}^+
\end{aligned}$$

Set

$$\begin{aligned}
\psi(y) &:= \sup_{\xi \in \mathbb{R}^n} L(\xi, y) \\
\phi(\xi) &:= \inf_{y>0} L(\xi, y) \ \text{ for } \xi \in \mathrm{dom}(U).
\end{aligned}$$

<u>Fact.</u>

$$\begin{aligned}
\sup_{\xi \in \mathrm{dom}(U)} \phi(\xi) &= u(x) \\
&= \sup_{\substack{\xi \in \mathrm{dom}(U) \\ \sum q_n \xi_n \leq x}} \sum q_n U(\xi_n)
\end{aligned}$$

Proof. Note that

$$\phi(\xi) = \begin{cases} -\infty & \text{if } \sum q_n \xi_n > x \\ \sum p_n U(\xi_n) & \text{if } \sum q_n \xi_n \leq x \end{cases}$$

∎

We expect that

$$u(x) = \sup_{\xi} \inf_{y>0} L(\xi, y) = \sup_{\xi} \phi(\xi)$$
$$(\text{also}) = \inf_{y>0} \sup_{\xi} L(\xi, y) = \inf_{y\geq 0} \psi(y)$$

We study ψ first:
1. We need to consider

$$\text{maximize } U(\xi) - y\frac{q_n}{p_n}\xi \text{ over } \xi \in \mathbb{R}.$$

Set $V(\eta) := \sup_{\xi} [U(\xi) - \eta\xi]$ (Legendre transform of U)
Then,

$$\psi(y) = \sum_n p_n V\left(y\frac{q_n}{p_n}\right) + yx$$
$$= \underbrace{\mathbb{E}^{\mathbb{P}} V\left(y\frac{d\mathbb{P}}{d\mathbb{Q}}\right)}_{:=v(y)} + yx$$

Theorem 9.10

$$u(x) := \sup_{X_T \in \mathscr{C}(x)} \mathbb{E}U(X_T),\ x \in dom(\mathrm{U})$$
$$v(y) := \mathbb{E}[V(y\frac{dP}{dQ})],\ \textit{for } y > 0.$$

Then,

1. *u and v are conjugate to each other, i.e.,*

$$u(x) = \sup_{y\geq 0} \{v(y) - yx\}$$
$$v(y) = \sup_{x\in dom(\mathrm{U})} \{u(x) - yx\};$$

2. *Maximizer $X_T^* \in \mathscr{C}(x)$ exists and is unique and*

$$X_T^* = I(y^*\frac{d\mathbb{Q}}{d\mathbb{P}}) \ \Leftrightarrow\ U'(X_T^*) = y^*\frac{d\mathbb{Q}}{d\mathbb{P}} \quad (I = (U')^{-1});$$

and $y^* > 0$ is given by

$$y^* = u'(x) \ \Leftrightarrow\ x = -v'(y^*);$$

3. $u'(x) = E[\mathrm{U}'(X_T^*)]$ & $v'(y) = \mathbb{E}^{\mathbb{Q}}[V'(y\frac{d\mathbb{P}}{d\mathbb{Q}})]$
$xu'(x) = \mathbb{E}[U'(X_T^*)X_T^*]$ & $yv'(y) = \mathbb{E}^{\mathbb{Q}}[y\frac{d\mathbb{P}}{d\mathbb{Q}}V'(y\frac{d\mathbb{P}}{d\mathbb{Q}})]$.

Note that V (conjugate of U) satisfies

$$V'(0) = \lim_{y \downarrow 0} V'(y) = -\infty,$$
$$V(0) = \lim_{y \downarrow 0} V(y) = U(\infty)$$

V is strictly convex on $(0, \infty)$

<u>Case 1:</u> $\lim_{y\to\infty} V(y) = U(0), \quad \lim_{y\to\infty} V'(y) = 0;$

<u>Case 2:</u> $\lim_{y\to\infty} V(y) = \infty, \quad \lim_{y\to\infty} V'(y) = \infty.$

Examples are,

- $U(x) = \ln x \ (x > 0), \quad V(y) = -\ln y - 1;$
- $U(x) = \frac{x^\alpha}{\alpha} \ (\alpha < 1, x > 0), \quad V(y) = \frac{1-\alpha}{\alpha} y^{\frac{\alpha}{\alpha-1}};$
- $U(x) = -\frac{e^{-\gamma x}}{\gamma} \ x \in \mathbb{R}, \quad V(y) = \frac{y}{\gamma}(\ln(y) - 1) \ \gamma > 0.$

Proof. Fix $x \in$dom(U) then there is $\hat{y}(x) > 0$ so that

$$\psi(\hat{y}(x)) = \min_{y>0} \left(E_P[V(y\frac{dQ}{dP}) + yx] \right)$$

Now consider the map

$$\xi \to L(\xi, \hat{y}(x)) = E_P \mathrm{U}(\xi) - \hat{y}(x)(E_Q(\xi) - x)$$

It has a unique maximum $\hat{\xi}(= \hat{\xi}(x)) \in \mathbb{R}^N$ and

$$\mathrm{U}'(\hat{\xi}_n) = \hat{y}(x)\frac{q_n}{p_n}, \ n = 1, ..., N$$
$$\Leftrightarrow \hat{\xi}_n = I\left(\hat{y}(x)\frac{q_n}{p_n}\right).$$

Then,

$$\begin{aligned}
\psi(\hat{y}(x)) &= \min_{y>0} \left(E^P[V(y\frac{dQ}{dP}) + yx] \right) \\
&= \min_{y>0} \ \sup_{\xi} E^P[\mathrm{U}(\xi) - y\frac{dQ}{dP}\xi] + yx \\
&= \min_{y>0} \ \sup_{\xi} L(\xi, y) = \sup_{\xi} L(\xi, \hat{y}(x)) = L(\hat{\xi}, \hat{y}(x)).
\end{aligned}$$

Easy to show that $\sum q_n \hat{\xi}_n = x \Rightarrow \qquad = E^P \mathrm{U}(\hat{\xi}) = u(x)$

$$\begin{aligned}
\inf_{y>0}\{v(y) + yx\} &= v(\hat{y}(x)) + \hat{y}(x)x \quad (definition of v) \\
&= E^P V(\hat{y}(x)\frac{dQ}{dP}) + \hat{y}(x)x \\
&= L(\hat{\xi}, \hat{y}(x)) \\
&= u(x).
\end{aligned}$$

Also

$$0 = v'(\hat{y}(x)) = \frac{\partial}{\partial y}[v(y) + yx]|_{y=\hat{y}(x)}$$

Also for any $t \in \mathbb{R}$

$$u(x+t) \leq v(\hat{y}(x)) + \hat{y}(x)(x+t)$$

with an equality at t=0. So differentiate with respect to t to get

$$u'(x) = \hat{y}(x).$$

(i) Now existence is proved

$$X_T^* = \hat{\xi} \text{ and } \hat{\xi} = I(\hat{y}(x)\frac{dQ}{dP}) \quad \Leftrightarrow \hat{y}(x)\frac{dQ}{dP} = \mathrm{U}'(\hat{\xi})$$

(ii) $u'(x) = \hat{y}(x) = E^P(\hat{y}(x)\frac{dQ}{dP}) = E^P\mathrm{U}'(\hat{\xi})$. Others proved similarly.

■

- Incomplete market

  $E_P\mathrm{U}(X_T) = \sum p_n \mathrm{U}(\xi_n)$ is maximized over all

$$E_{Q_m}[X_T] = \sum q_n^m \xi_n \leq x \quad \forall Q_m \in \mathcal{M}$$
$$\mathcal{M} = \{Q = (q_1, ..., q_n) \ : \ \sum q_n = 1, \ 0 \leq q_n \leq 1, \ E_N^Q S_m = S_N\}$$

  Note that $\mathcal{M}$ is a convex, bounded subset of $\mathbb{R}^n$ and it is a polytope as it defined through finitely manu kinear equations. Hence,

$$\mathcal{M} = \overline{co}\{Q_1, ..., Q_m\}$$

  We now define

$$\begin{aligned} L(\xi,\eta) &:= \sum p_n \mathrm{U}(\xi_n) - \sum_{m=1}^{M} \eta_m \left[\sum_{n=1}^{N} q_n^m \xi_n - x\right] \\ &= \sum p_n \left[\mathrm{U}(\xi_n) - \sum_{m=1}^{M} \frac{\eta_m q_n^m}{p_n} \xi_n\right] + x(\sum \eta_m) \end{aligned}$$

  Set

$$\mu := \frac{\eta}{y}, \quad y = \sum \eta_m, \quad Q^M := \sum \mu_m Q^m$$

  Redefine

$$L(X_T, Q, y) := E_P(\mathrm{U}(X_T)) - yE_Q(X_T - x)$$
$$\Leftrightarrow L(\xi, y, q) := \sum p_n \left(\mathrm{U}(\xi_n) - \frac{yq_m}{p_n}\xi_n\right) + yx \quad q = (q_1, ..., q_n) \in \mathcal{M}$$

Again

$$\begin{aligned}
\psi(y,Q) &:= \sup_{\xi} \; L(\xi,y,Q) \\
&= \sum_{n=1}^{N} p_n V(\frac{yq_n}{p_n}) + yx \\
\psi(y) &:= \inf_{Q\in\mathcal{M}} \; \psi(y,Q) \\
v(y) &:= \inf_{Q} \sum p_n V\left(y\frac{q_n}{p_n}\right) = \sum p_n V\left(y, \frac{\hat{q}_n(y)}{p_n}\right)
\end{aligned}$$

Then,

$$\xi^* = I\left(\hat{y}(x)\frac{\hat{q}_n(y)}{p_n}\right)$$

- Conclusion.

$$\begin{aligned}
u(x) &= \sup_{X_T\in\mathscr{C}(x)} \; E\mathrm{U}(X_T); \quad x\in \mathrm{dom}\mathrm{U} \\
v(y) &:= \inf_{Q\in\mathcal{M}} E\left[V\left(y\frac{dQ}{dP}\right)\right] \quad y>0
\end{aligned}$$

1. u and v are conjugate to each other;
2. $\hat{X}_T(x)$ and $\hat{Q}(y)$ exists and

$$y\frac{\hat{Q}(y)}{dP} = \mathrm{U}'(\hat{X}_T(x))$$

Again a unique martingale measure is chosen by this procedure.

- Continuous Time.

  Assume $\lim_{x\uparrow\infty} \frac{x\mathrm{U}'(x)}{\mathrm{U}} < 1$ (asymptotic elasticity < 1)

  case 2 $\lim_{x\downarrow-\infty} \frac{x\mathrm{U}'(x)}{\mathrm{U}} > 1$

  Assume $\mathrm{dom}\mathrm{U} = \mathbb{R}_+$

  Min-max theorems are central to reasoning.

- The ∞-dimensional version of thm : (Ekeland and Temam '76)

  E, F be a pair of locally convex vector spaces in duality.

  $C\subseteq E$, $D\subseteq F$ are convex sets, $L: C\times D\to\mathbb{R}$ concave on C, convex on D and has some semi-continuity. If C/D compact/complete, then $\exists\ \hat{\xi}\in C, \hat{\eta}\in D$ so that

$$L(\hat{\xi},\hat{\eta}) = \sup_{\xi\in C} \; \inf_{\eta\in D} \; L(\xi,\eta) = \inf \; \sup \; L(\xi,\eta).$$

  In our application

$$L(X_T, Q) = E_P \mathrm{U}(X_T) - yE_Q(X_T) + yx$$

$X_T \geq 0$ $\mathcal{F}_T$-mbl and $Q \in \mathcal{M}^q$. Set

$$\begin{aligned} C(x) &= \{X_T \in L^0(\mathcal{F}_T) \ : \ X_t \geq 0, \ X_T \leq x + \textstyle\int H_u dS_u\} \\ &= \{X_T \ : \ E_Q(X_T) \leq x, \quad \forall Q \in \mathcal{M}^q(S)\} \\ D &:= \{Y_T \in L^0(\mathcal{F}_T) \ : \ Y_T \geq 0 \quad \exists Q_n \in \mathcal{M}^q \quad Y_T \leq \lim_{n \to \infty} \frac{dQ_n}{dP}\} \end{aligned}$$

Then

$$X_T \in C \Leftrightarrow \ E_P(X_T Y_T) \leq 1 \quad \forall Y_T \in D$$

We also have (bipolar thm)

$$! \ Y_T \in D \ \Leftrightarrow E_P(X_T Y_T) \leq 1 \quad \forall X_T \in C!$$

Then continue with the usual convex analysis. Note that the existence of $Q \in \mathcal{M}^q$ is no longer guaranteed.