ETH Zürich HS 2015
D-MATH
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Mathematical Finance
Exercise Sheet 3

Please hand in until Friday, 23.10.2015, 12:00.

Exercise 3-1

Consider a multidimensional Ito process model with finite time horizon $T < \infty$, where for any $i = 1, \dots, d$ the dynamics of the ith stock price process are given by

$$\begin{aligned} d\tilde{S}_t^i &= \tilde{S}_t^i \mu_t^i dt + \tilde{S}_t^i \sum_{j=1}^n \sigma_t^{ij} dW_t^j, \\ \tilde{S}_0^i &= s^i > 0. \end{aligned}$$

Here $W = (W^1, \dots, W^n)$ denotes a standard n-dimensional Brownian motion and $\{\mathcal{F}_t\}_{0 \le t \le T}$ is the filtration generated by W satisfying the usual conditions. The coefficients σ and μ are assumed to be bounded predictable processes with values in $\mathbb{R}^{d \times n}$ and $\mathbb{R}^d$ respectively. The bank account is modelled by

$$\begin{aligned} d\tilde{B}_t &= \tilde{B}_t r_t dt \\ \tilde{B}_0 &= 1. \end{aligned}$$

We suppose furthermore that there exists an ELMM Q for the discounted stock price process. Assume now that the number of stocks d is equal to the dimension n of the underlying Brownian motion and that the volatility matrix $\sigma = (\sigma^{ij})$ is $dP \otimes dt$-a.s. nonsingular. Prove that these conditions imply the following:

a) The market price of risk λ is $dP \otimes dt$-a.s. unique.

b) ELMM Q is unique.

c) Every discounted payoff $H \in L^\infty(\mathcal{F}_T)$ is attainable (with an admissible self-financing strategy). Hint: Consider the Q-martingale with final value H and use a martingale representation theorem under P.

Exercise 3-2

Assume Black-Scholes framework with interest rate r, and time running from 0 to T. Consider the option with payoff $h(S_T) = S_T^p$ at time T, for some $p \in \mathbb{R}$. Calculate the corresponding hedging strategy $\varphi_t = (\eta_t, \theta_t)$.

Exercise 3-3

Let $T > 0$ be finite time horizon, and an adapted cadlag $U = (U_t)_{0 \le t \le T}$, with $\sup_{t \in [0,T]} |U_t| \in L^1$, be the discounted exercise value of an American option. Under the risk neutral pricing probability, the selling price of the American option is given by

$$V_t = \operatorname{ess\,sup}_{t \le \tau \le T} E[U_\tau \mid \mathcal{F}_t],\ t \in [0, T].$$

Show that

$$V_0 = \inf_{M \in H_0^1} E[\sup_{0 \le t \le T} (U_t - M_t)],$$

where H_0^1 is the space of martingales M for which $\sup_{t \in [0,T]} |M_t| \in L^1$, and $M_0 = 0$. Assume $V_0 < \infty$.

Exercise 3-4

Let $w \in C^2([0, \infty])$. Show that under the risk neutral pricing probability

$$E[w(S_T)] = w(S_0) + \int_0^{S_0} w''(K)P(K)dK + \int_{S_0}^{\infty} w''(K)C(K)dK,$$

where $S_0 = E[S_T]$, $C(K) = E[(S_T - K)^+]$, and $P(K) = E[(K - S_T)^+]$.

Definition 1 Given a topological vector space X, its dual X^*, and a convex proper (i.e. $-\infty < f \not\equiv \infty$) extended real valued f on X, let us recall that the sub-differential mapping $\partial f : X \to 2^{X^*}$ is

$$\partial f(x) := \{x^* \in X^* : f(y) \ge f(x) + x^*(y - x)\ \forall y \in X\}$$

on $\operatorname{dom}(f) := \{x \in X : f(x) \in \mathbb{R}\}$, and $\emptyset$ otherwise.

Exercise 3-5

Let f_n and f be convex proper functions on $\mathbb{R}^d$ such that $f_n \to f$ locally uniformly on $\operatorname{dom}(f)$. Let $x_n \to x \in \operatorname{int}\operatorname{dom}(f)$, and $x_n^* \in \partial f_n(x_n)$. Show that (x_n^*) is bounded and any accumulation point of (x_n^*) is in $\partial f(x)$.