Lecture 7: Convex Analysis and Fenchel-Moreau Theorem

The main tools in mathematical finance are from theory of stochastic processes because things are random. However, many objects are convex as well, e.g. collections of probability measures or trading strategies, utility functions, risk measures, etc.. Convex duality methods often lead to new insight, computational techniques and optimality conditions; for instance, pricing formulas for financial instruments and characterizations of different types of no-arbitrage conditions.

Convex sets

Let X be a real topological vector space, and X^* denote the topological (or algebraic) dual of X. Throughout, we assume that subsets and functionals are proper, i.e., $\emptyset \neq C \neq X$ and $-\infty < f \not\equiv \infty$.

Definition 1. Let $C \subset X$. We call C affine if

$$\lambda x + (1-\lambda)y \in C \;\forall x, y \in C,\; \lambda \in]-\infty, \infty[,$$

convex if

$$\lambda x + (1-\lambda)y \in C \;\forall x, y \in C,\; \lambda \in [0,1],$$

cone if

$$\lambda x \in C \;\forall x \in C,\; \lambda \in]0, \infty].$$

Recall that a set A is called algebraic open, if the sets $\{t : x + tv\}$ are open in $\mathbb{R}$ for every $x, v \in X$. In particular, open sets are algebraically open.

Theorem 1. *(Separating Hyperplane Theorem) Let A, B be convex subsets of X, A (algebraic) open. Then the following are equal:*

1. *A and B are disjoint.*
2. *There exists an affine set $\{x \in X : f(x) = c\}$, $f \in X^*$, $c \in \mathbb{R}$, such that $A \subset \{x \in X : f(x) < c\}$ and $B \subset \{x \in X : f(x) \geq c\}$.*

If in addition A and B are cones, we may take $c = 0$.

Proof. 2) $\implies$ 1) is obvious, 1) $\implies$ 2) exercise. □

On a locally convex X, Separating Hyperplane Theorem gives an useful characterization of closed convex sets:

$$A \subset X \text{ is closed and convex} \iff A = \bigcap_{\alpha} \{x \in X : f_\alpha(x) \geq \alpha,\; f_\alpha \in X^*,\; \alpha \in \mathbb{R}\}.$$

If X is also separable, then one can replace $\mathbb{R}$ with $\mathbb{N}$ above.

Remark 1. *Let X be a d-dimensional Hilbert space ($d = \#$ of base vectors). We say that $C \subset X$ is Chebyshev set if for every $x \in X$ there exists unique $\tilde{y} \in C$ such that $\tilde{y} = \arg\min_{y \in C} ||x - y||$. Consider the following statement:*

$$C \text{ is Cheyshev set} \iff C \text{ is closed and convex.}$$

If $d < \infty$, then it is an easy exercise to verify that the statement above is true. However, for $d = \infty$, this an open problem (convexity in the implication "$\Rightarrow$"). Lesson: Be careful when $d = \infty$.

Convex functionals

Definition 2. We call f affine on X if

$$\lambda f(x) + (1 - \lambda) f(y) = f(\lambda x + (1 - \lambda y)) \; \forall x, y \in X, \; \lambda \in] - \infty, \infty[,$$

convex if

$$\lambda f(x) + (1 - \lambda) f(y) \leq f(\lambda x + (1 - \lambda) y) \; \forall x, y \in X, \; \lambda \in [0, 1],$$

positively homogeneous if

$$f(\lambda x) = \lambda f(x) \; \forall x \in X, \lambda \in]0, \infty],$$

sub-additive if

$$f(x + y) \leq f(x) + f(y) \; \forall x, y \in X.$$

Remark 2. *If f is convex and positively homogeneous, then f is sub-additive. Positively homogeneous and sub-additive f is called sub-linear.*

Theorem 2. *(Hahn-Banach Theorem) Let Y be a subspace of X, and f linear functional on Y. If there exists a sub-linear functional g on X such that $f \leq g$ on Y (and g continuous at 0), then there exists $\tilde{f} \in X^*$ such that $\tilde{f} = f$ on Y and $f \leq g$ on X.*

Proof. This is proven at the basic course of functional analysis. □

Remark 3. *Separating Hyperplane Theorem (SHT) and Hahn-Banach Theorem (HBT) are equal in the following sense: If we assume that SHT is true, then HBT is true. Conversely, if we assume that HBT is true, then SHT is true.*

Convex conjugates

We make an additional assumption that X is Hausdorff and locally convex.

Definition 3. The epigraph of f is defined as

$$\operatorname{epi} f := \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}.$$

If $\operatorname{epi} f$ is closed, we say that f is lower semicontinuous (abbreviated l.s.c.).

Lemma 3. *If f is l.s.c. and convex, then, for every $x \in X$,*

$$f(x) = \sup_{a \leq f} a(x),$$

where the supremum is taken over all continuous affine functionals on X.

Proof. The idea of proof: "If a point does not belong to the epigraph, then there is an affine minorant in between."

Let $(x_0, \alpha_0) \in X \times \mathbb{R} \setminus \operatorname{epi} f$. By Separating Hyperplane Theorem, there exists $(x^*, c) \in X^* \times \mathbb{R}$ such that

$$x^*(x) + \alpha c > x^*(x_0) + \alpha_0 c, \ \forall (x, \alpha) \in \operatorname{epi} f.$$

If $c \neq 0$, it can be scaled to $c = 1$. Then $\inf_{(x,\alpha)\in \operatorname{epi} f}\{x^*(x) + \alpha\} - x^*(x)$ is an affine minorant of f whose epigraph does not contain (x_0, α_0). So, assume $c = 0$. Choose $x_1 \in X$ such that $f(x_1) < \infty$. Then $(x_1, f(x_1) - 1) \notin \operatorname{epi} f$, and, by Separating Hyperplane Theorem, there exists $(y^*, c') \in X^* \times \mathbb{R}$ such that

$$y^*(x) + \alpha c' > y^*(x_1) + (f(x_1) - 1)c', \ \forall (x, \alpha) \in \operatorname{epi} f.$$

Since $f(x_1) < \infty$, we have $c' \neq 0$, and, by scaling, we can assume $c' = 1$. Choosing

$$\delta > \frac{y^*(x_0) + \alpha_0 - \inf_{(x,\alpha)\in \operatorname{epi} f}\{y^*(x) + \alpha\}}{\inf_{(x,\alpha)\in \operatorname{epi} f}\{x^*(x) + \alpha\} - x^*(x_0)}$$

and setting $z^* = \delta x^* + y^*$ yields

$$\inf_{(x,\alpha)\in \operatorname{epi} f}\{z^*(x)+\alpha\} \geq \delta \inf_{(x,\alpha)\in \operatorname{epi} f}\{x^*(x)+\alpha\} + \inf_{(x,\alpha)\in \operatorname{epi} f}\{y^*(x)+\alpha\} > z^*(x_0)+\alpha_0.$$

So, $\inf_{(x,\alpha)\in \operatorname{epi} f}\{z^*(x) + \alpha\} - z^*(x)$ is an affine minorant of f whose epigraph does not contain (x_0, α_0). □

Definition 4. Let $f : X \to \mathbb{R} \cup \{\pm\infty\}$. Then $f^* : X^* \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$f^*(x^*) := \sup_{x \in X}\{x^*(x) - f(x)\}$$

is called the convex conjugate of f, and the mapping

$$f \mapsto f^*$$

is called Legendre-Fenchel transformation.

Theorem 4. *(Fenchel-Moreau Theorem) If f is l.s.c. and convex, then Legendre-Fenchel transformation is bijection: $f^{**} = f$, where*

$$f^{**}(x) := \sup_{x^* \in X^*}\{x^*(x) - f^*(x^*)\}.$$

Proof. By the definition of f^*,

$$f(x) \geq \sup_{x^* \in X^*} \{x^*(x) - f^*(x)\}, \text{ for every } x \in X.$$

i.e. $f \geq f^{**}$. Let a be an affine minorant of f; $a \leq f$, so, $a^* \geq f^*$, so, $a^{**} \leq f^{**}$, but since a is affine, $a^{**} = a$. So, every affine minorant of f is an affine minorant of f^{**}. By Lemma 3, $f \leq f^{**}$. □

We could have proved the theorem also making the following observation: for fixed $x \in X$, the directional derivative

$$f'(x; y) := \lim_{\epsilon \downarrow 0} \frac{f(x + \epsilon y) - f(x)}{\epsilon}$$

is sub-linear as a functional of y. By Hahn-Banach Theorem, there exists $\tilde{f}' \in X^*$ such that

$$\tilde{f}'(y) \leq f'(x; y) \leq f(x + y) - f(x), \text{ for every } y \in X.$$

So, $f(x) + f^*(\tilde{f}') = \tilde{f}'(x)$, which completes the proof. We say that $\tilde{f}'$ is a sub-gradient of f at x, and denote $\tilde{f}' \in \partial f(x)$. The collection of all sub-gradients ∂f is called sub-differential.

References

These lecture notes were written in a hurry and may contain erors. See sources listed below for details and full treatment of the things presented and discussed on the lecture.[Che13],[Roc70],[RWW98],[Pen12]

References

[Che13] Patrick Cheridito. Convex analysis, lecture notes, 2013.

[Pen12] T. Pennanen. Introduction to convex optimization in financial markets. *Math. Program.*, 134(1, Ser. B):157–186, 2012.

[Roc70] R. T. Rockafellar. *Convex analysis.* Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[RWW98] Ralph Tyrrell Rockafellar, Roger J.-B Wets, and Maria Wets. *Variational analysis.* Grundlehren der mathematischen Wissenschaften. Springer, Berlin, Heidelberg, New York, 1998. Autres tirages : 2004.