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Sheet 2

1. Let q denote the function from $\mathbb{H}$ to the punctured open unit ball defined by $q(z) := e^{2\pi i z}$.

The function $\Delta : \mathbb{H} \to \mathbb{C}$ that sends $z \in \mathbb{H}$ to $\Delta(z) := (2\pi)^{12} q(z) \prod_{n=1}^{\infty}(1 - q(z)^n)^{24}$ is called the *discriminant function*.

Moreover, the function $\tau : \mathbb{N}_{>0} \to \mathbb{Z}$ defined trough the series

$$\sum_{n=1}^{\infty} \tau(n) q^n := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 + O(q^4) \in \mathbb{Z}[[q]]$$

is called the *Ramanujan τ-function*.

a) Show that $\frac{1}{2\pi i} \frac{d}{dz} log\Delta(z) = E_2(z)$ and conclude that $\Delta \in S_{12}$.

b) Show that $\Delta = \frac{(2\pi)^{12}}{1728} \left(E_4^3 - E_6^2\right)$ and derive relations expressing τ in terms of σ_3 and σ_5.

c) Show that $E_{12} - E_6^2 = c\Delta$ with $c = (2\pi)^{-12} 2^6 3^5 7^2 \frac{1}{691}$ and derive relations expressing τ in terms of σ_{11} and σ_5. Use this to prove Ramanujan's famous congruence relation

$$\tau(n) \equiv \sigma_{11}(n) \mod 691$$

for any $n \geq 1$.

2. **a)** Show that $E_4^2 = E_8$ and $E_4 E_6 = E_{10}$ and $E_6 E_8 = E_{14}$ and derive relations expressing

- σ_7 in terms of σ_3
- σ_9 in terms of σ_3 and σ_5
- σ_{13} in terms of σ_5 and σ_7.

b) Let $f \in M_k$ and set

$$g(z) := \frac{1}{2\pi i} \frac{d}{dz} f(z) - \frac{k}{12} E_2(z) f(z)$$

for any $z \in \mathbb{H}$. Show that this defines a modular form $g \in M_{k+2}$ and that $g \in S_{k+2}$ if and only if $f \in S_k$.

c) Compute g explicitly for the cases where f is Δ or E_4 or $f = E_6$ and derive relations expressing

- σ_5 in terms of σ_1 and σ_3

- σ_7 in terms of σ_1 and σ_5.

3. We set $\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\} \subset \mathrm{SL}_2(\mathbb{Z})$. This is the stabilizer of infinity in any congruence subgroup of the form $\Gamma_0(N)$ and $\Gamma_1(N)$, where $N \geq 1$.

a) Find a system of representatives for $\Gamma_\infty \backslash \Gamma_0(N)$ for any $N \geq 1$.

b) Let $k \geq 4$ be an even integer and $N \geq 1$. For any $z \in \mathbb{H}$ we set

$$E_{k,N}(z) := \sum_{[\gamma] \in \Gamma_\infty \backslash \Gamma_0(N)} 1|[\gamma]_k(z) = \sum_{[\gamma] \in \Gamma_\infty \backslash \Gamma_0(N)} j(\gamma, z)^{-k} \quad \text{and}$$

$$G_{k,N}(z) := \frac{1}{2} \sum_{m,n \in \mathbb{Z}}{}' \frac{1}{(Nmz + n)^k}.$$

Show that this this defines modular forms in $M_k(\Gamma_0(N))$ with $G_{k,N} = \zeta(k) E_{k,N}$.

The modular form $E_{k,N}$ is called the *(normalised) Eisenstein series* of weight k and level N.

c) Let $k \geq 0$. For any positive integers N and M such that N divides M we consider the *trace operator*

$$tr_N^M : M_k(\Gamma_0(M)) \to M_k(\Gamma_0(N))$$

defined by

$$tr_N^M(f) := \sum_{[\gamma] \in \Gamma_0(M) \backslash \Gamma_0(N)} f|[\gamma]_k$$

for any $f \in M_k(\Gamma_0(M))$.

Check that it is indeed well-defined and maps $S_k(\Gamma_0(M))$ to $S_k(\Gamma_0(N))$.

Show moreover that $tr_N^M(G_{k,M}) = G_{k,N}$ and in particular, that $tr_1^M(G_{k,M}) = G_k$.

4. **a)** Prove that

$$\frac{\sin(z)}{z} = \prod_{n \geq 1} (1 - \frac{z^2}{n^2 \pi^2})$$

for any $z \in \mathbb{C}$.

Hint: Use that

$$\frac{\sin(z)}{z} = \frac{e^{iz} - e^{-iz}}{2iz} = \lim_{n \to \infty} p_n(z),$$

where

$$p_n(z) := \frac{(1 + \frac{iz}{n})^n - (1 - \frac{iz}{n})^n}{2iz}$$

and show that

$$p_n(z) = \prod_{1 \leq i \leq p} \left(1 - \frac{z^2}{n^2} \left(\frac{1 + \cos(\frac{2k\pi}{n})}{1 - \cos(\frac{2k\pi}{n})} \right) \right)$$

whenever $n = 2p + 1$ for some integer p.

b) Use logarithmic differentiation in order to deduce Euler's identity

$$\pi\cot(\pi z) = \frac{1}{z} + 2z\sum_{n\geq 1}\frac{1}{z^2-n^2}.$$

c) Compare the power series expansions of $z\pi\cot(\pi z)$ and of $\frac{z}{e^z-1} = \sum_{k\geq 0}\frac{B_k z^k}{k!}$ in order to prove Euler's formula

$$\zeta(2k) = \frac{(-1)^{k-1}2^{2k-1}B_{2k}}{(2k)!}\pi^{2k}$$

for the value of the Riemann zeta function ζ at any positive even integer $2k$.

Conclude that $\frac{\zeta(2k)}{\pi^{2k}}$ is rational.