Sheet 4

1. Let N be any positive integer and χ be any non-tivial Dirichlet character modulo N. Let a be any integer. Then the sum

$$G(a,\chi) := \sum_{1\leq m\leq N} \chi(m)\xi_N^{am},$$

where $\xi_N := e^{\frac{2\pi i}{N}}$, is called the *Gauss sum* associated with χ and a.

Show the following statements.

a) If $\gcd(a,N) = 1$, then $G(a,\chi) = \overline{\chi}(a)G(1,\chi)$.

b) We have $G(a,\chi) = \overline{\chi}(a)G(1,\chi)$ for any integer a if and only if we have $G(a,\chi) = 0$ for any integer a with $\gcd(a,N) > 1$.

c) If $G(a,\chi) = \overline{\chi}(a)G(1,\chi)$ for any integer a, then

$$|G(1,\chi)|^2 = N. \tag{1}$$

In particular, (1) holds if N is prime.

2. Let χ be any non-trivial even Dirichlet character modulo a prime p. Define

$$\Theta(\chi,t) := \sum_{1\leq n}\chi(n)e^{-\pi tn^2} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\chi(n)e^{-\pi tn^2}$$

for any real $t > 0$. For any real α and any real $t > 0$ define

$$\Theta_\alpha(t) := \sum_{n\in\mathbb{Z}} e^{-\pi t(n+\alpha)^2} \quad \text{and} \quad \Theta^\alpha(t) := \sum_{n\in\mathbb{Z}} e^{2\pi n\alpha}e^{-\pi tn^2}.$$

a) Show that

1. $\Theta(\chi,t) = \frac{1}{2}\sum_{1\leq a\leq p}\chi(a)\Theta_{\frac{a}{p}}(p^2t)$,
2. $\frac{1}{2}\sum_{1\leq a\leq p}\chi(a)\Theta^{\frac{a}{p}}(t) = G(1,\chi)\Theta(\overline{\chi},t)$,
3. $\Theta(\chi,t) = \frac{G(1,\chi)}{\sqrt{p^2t}}\Theta(\overline{\chi},\frac{1}{p^2t})$.

b) Let $L(\chi,s) := \sum_{n\geq 1}\frac{\chi(n)}{n^s}$ be the L-series associated with χ. Show that for any $s\in\mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}$ we have

$$\pi^{-s}\Gamma(s)L(\chi,2s) = \int_0^\infty \Theta(\chi,t)t^s\frac{dt}{t}.$$

c) Show that $L(\chi, \cdot)$ has an analytic continuation to $\mathbb{C}$ which is entire and find a functional equation that relates $L(\chi, s)$ with $L(\overline{\chi}, 1-s)$.

3. Let $\Gamma_\vartheta := \langle T^2, S\rangle = \langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle \subset SL_2(\mathbb{Z})$.

a) Show that

$$\begin{aligned}\Gamma_\vartheta &= \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mod 2 \right\} \\ &= \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, a+b+c+d \equiv 0 \mod 2 \right\} \\ &= \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, ab \equiv cd \equiv 0 \mod 2 \right\}.\end{aligned}$$

In particular, $\Gamma(2) \subset \Gamma_\vartheta \subset SL_2(\mathbb{Z})$. Hence Γ_ϑ is a congruence subgroup of level 2.

b) Show that

$$SL_2(\mathbb{Z}) = \Gamma_\vartheta \,\dot\cup\, \Gamma_\vartheta T \,\dot\cup\, \Gamma_\vartheta TS$$

and draw a picture of a fundamental domain of Γ_ϑ. Show that Γ_ϑ has the two cusps $[i\infty]$ and $[1] = TS[i\infty]$.

c) Let k be an integer. Recall that a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is in $M_k(\Gamma_\vartheta)$ if and only if

1. $f|_k T^2 = f$,
2. $f|_k S = f$,
3. the limit $f(i\infty) := \lim_{y\to\infty} f(\tau)$ exists,
4. the limit $f(1) := \lim_{y\to\infty} (f|_k TS)(\tau) = \lim_{y\to\infty} \tau^{-k} f\left(1 - \frac{1}{\tau}\right)$ exists.

Show that there are no non-constant modular forms of weight zero for Γ_ϑ, i.e. that $M_0(\Gamma_\vartheta) = \mathbb{C}$.

4. For any natural numbers $r \geq 1$ and $n \geq 0$ consider the number

$$A_r(n) := |\{x = (x_1, ..., x_r) \in \mathbb{Z}^r \mid x_1^2 + ... + x_r^2 = n\}|$$

of possibilities to write n as a sum of r squares of integers. The goal of this exercice is to prove Jacobi's 8-squares formula

$$A_8(n) = 16 \sum_{d|n} (-1)^{n-d} d^3.$$

To this end consider the theta-series $\vartheta(\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}$ and $\Theta(\tau) := \vartheta(2\tau)$ so that

$$\Theta^r(\tau) = \sum_{n=0}^{\infty} A_r(n) e^{2\pi n \tau}.$$

a) Show that ϑ satisfies

1. $\vartheta(\tau+2) = \vartheta(\tau)$,
2. $\vartheta(\frac{-1}{\tau}) = \sqrt{\frac{\tau}{i}}\vartheta(\tau)$,
3. $\lim_{y\to\infty} \vartheta(\tau) = 1$,
4. $\lim_{y\to\infty} \left(\sqrt{\frac{\tau}{i}}^{-1}\vartheta(1-\frac{1}{\tau})e^{-\pi i \frac{\tau}{4}}\right) = 2$.

In particular, $\vartheta(i\infty) = 1$ and $\vartheta(1) = 0$.

b) Show that ϑ does not vannish on $\mathbb{H} \cup \{i\infty\}$.

c) Suppose that $r \equiv 0$ modulo 4. Let $f \in M_{\frac{r}{2}}(\Gamma_\vartheta)$ such that

$$\lim_{y\to\infty} \left(\left(\sqrt{\frac{\tau}{i}}\right)^{-r} f(1-\frac{1}{\tau})e^{-\pi i \frac{\tau r}{4}} \right)$$

exists. Note that this condition is stronger than the condition at the cusp [1] in the Definition of $M_{\frac{r}{2}}(\Gamma_\vartheta)$. Show that then $f = c\vartheta^r$ for some constant $c \in \mathbb{C}$.

Hint: Use the last part of Exercise 3.

d) Consider the Eisenstein series

$$G_k(\tau) := \sum_{(0,0)\neq(c,d)\in\mathbb{Z}^2} \frac{1}{(c\tau+d)^k}$$

and consider any $a, b \in \mathbb{C}$. Show that $f(\tau) := aG_k(\tau) + bG_k(\frac{\tau+1}{4}) \in M_k(\Gamma_\vartheta)$. Moreover, show that if $k = 4$ and $a = -16b$, then $f = c\vartheta^8$ for some $c \in \mathbb{C}$.

e) Show that

$$\vartheta^8 = \frac{3}{\pi^4}\left(16G_4(\tau) - G_4(\frac{\tau+1}{2})\right)$$

and conclude that

$$A_8(n) = 16\sum_{d|n}(-1)^{n-d}d^3.$$