Sheet 6

1. Set $\Gamma := SL_2(\mathbb{Z})$ and let $\alpha \in GL_2^+(\mathbb{Q})$.

a) Show that the subgroup $\Gamma_1 := \alpha^{-1}\Gamma\alpha \cap \Gamma$ is a congruence subgroup.

b) Show that for any $f, g \in S_k(\Gamma)$ we have that $f|_k\alpha, g|_k\alpha \in S_k(\Gamma_1)$ and that

$$\langle f, g\rangle = \langle f|_k\alpha, g|_k\alpha\rangle,$$

where the latter inner product is with respect to Γ_1.

2. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, fix $\alpha \in GL_2^+(\mathbb{Q})$ and set $\Gamma_1 := \alpha^{-1}\Gamma\alpha \cap \Gamma$.

a) Consider the double coset $\Gamma\alpha\Gamma$ in $\mathrm{GL}_2^+(\mathbb{Q})$. There is a natural group action by Γ on $\Gamma\alpha\Gamma$ by left matrix multiplication. Show that the assignment

$$\Gamma \ \to \ \Gamma\alpha\Gamma, \quad \gamma \mapsto \alpha\gamma$$

yields a one-to-one correspondence between $\Gamma\backslash\Gamma\alpha\Gamma$ and $\Gamma_1\backslash\Gamma$. By part **a)**, it follows that the orbit space $\Gamma\backslash\Gamma\alpha\Gamma$ is finite.

Let the slash operator for $\mathrm{GL}_2^+(\mathbb{Q})$ be defined as

$$f|_k[\gamma](z) \ = \ (\det\gamma)^{k-1} j(\gamma, z)^{-k} f(\gamma z)$$

for any $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$. Let $f \in \mathcal{M}_k(\Gamma)$. Consider the double coset slash operator given by

$$f|_k[\Gamma\alpha\Gamma] \ := \sum_{[\gamma]\in\Gamma\backslash\Gamma\alpha\Gamma} f|_k[\gamma].$$

b) Check that the above double coset operator is well-defined. Then show that the assignment $f \ \mapsto \ f|_k[\Gamma\alpha\Gamma]$ defines an operator

$$\mathcal{M}_k(\Gamma) \ \to \ \mathcal{M}_k(\Gamma), \quad \text{resp.} \quad \mathcal{S}_k(\Gamma) \ \to \ \mathcal{S}_k(\Gamma).$$

3. We set $\Gamma := SL_2(\mathbb{Z})$ and let k be any integer. Let p be a prime number and consider

$$\alpha_p := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \in SL_2(\mathbb{Z}).$$

The p'th *Hecke operator* of weight k is the operator $T_p : M_k(\Gamma) \to M_k(\Gamma)$ that sends $f \in M_k(\Gamma)$ to $T_p(f) := p^{\frac{k}{2}-1} f|_k[\Gamma\alpha_p\Gamma]$.

a) Show that

$$\Gamma\alpha_p\Gamma = \{\alpha \in GL_2(\mathbb{Z}) \mid \det \alpha = p\} = \bigcup_{j=0}^{p-1} \Gamma \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \dot{\cup}\, \Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

and conclude that

$$T_p(f)(\tau) = \frac{1}{p}\sum_{j=0}^{p-1} f\left(\frac{\tau+j}{p}\right) + p^{k-1} f(p\tau)$$

for any $f \in M_k(\Gamma)$. This shows that the definition of the Hecke operator T_p given here agrees with the one given in class.

b) For any $f = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma)$ consider

$$V_m(f) := \sum_{n=0}^{\infty} a_n q^{mn}$$

and

$$U_m(f) := \sum_{n=0}^{\infty} a_m n q^n.$$

Show that

$$T_p(f) = U_p(f) + p^{k-1} V_p(f) = \sum_{n=0}^{\infty} b_n q^n,$$

where $b_n := a_{pn} + p^{k-1} a_{\frac{n}{p}}$ and $a_{\frac{n}{p}} := 0$ if p does not divide n.

c) Show that

$$1 - T_p X + p^{k-1} X^2 = (1 - U_p X)(1 - p^{k-1} V_p X)$$

where both sides are regarded as polynomials in the variable X with coefficients in the Hecke algebra $\mathscr{H}$ which operates on the $\mathbb{C}$-subspace of $\mathbb{C}[[q]]$ formed by the q-expansions of elements $f \in M_k$.

Prove that the following formal identities hold:

$$\sum_{n=1}^{\infty} T_n n^{-s} = \left(\sum_{n=1}^{\infty} n^{k-1} V_n n^{-s}\right)\left(\sum_{n=1}^{\infty} U_n n^{-s}\right) \qquad T_n = \sum_{d|n} d^{k-1} V_d U_{\frac{n}{d}}$$

4. a) Let $k \geq 4$ be an even integer and let d be the $\mathbb{C}$-dimension of $S_k(SL_2(\mathbb{Z}))$. Choose any non-negative integers a, b such that $12 \neq 4a + 6b \leq 14$ and $4a + 6b = k \mod 12$. For each $1 \leq j \leq d$, define

$$f_j := \Delta^j E_6^{2(d-j)+b} E_4^a = \sum_{1 \leq n < \infty} a_n^{(j)} q^n,$$

where Δ is the normalized discriminant function and E_4 and E_6 are the normalized Eisenstein series of weight 4 respectively 6. Verify that $a_n^{(j)} = 0$ for $n < j$ and $a_j^{(j)} = 1$. Conclude that the f_j form a basis for $S_k(SL_2(\mathbb{Z}))$. This basis is called the *Miller basis*. Show moreover, that a modular form in $S_k(SL_2(\mathbb{Z})$ has integral Fourier coefficients if and only if it is a $\mathbb{Z}$-linear combination of the Miller basis.

b) Let f be a normalized Hecke eigenform for $SL_2(\mathbb{Z})$. Show that the Fourier coefficients of f are algebraic integers.