Sheet 8

1. Let $D < 0$ be a fundamental discriminant. Let χ_D be the Kronecker character modulo $|D|$. In the lecture it was proved the class number formula

$$L(\chi_D, 1) = \frac{2\pi h}{\omega\sqrt{|D|}}, \tag{1}$$

where h is the class number of $\mathbb{Q}(\sqrt{D})$ and ω is the number of roots of unity in $\mathbb{Q}(\sqrt{D})$.

The goal of this exercise is to prove using (1) that

$$h = \frac{-\omega}{2|D|}\sum_{n=1}^{|D|} n\chi_D(n).$$

We follow Stark [1].

a) Let k be a positive integer and $\chi : \mathbb{Z} \to \mathbb{Z}$ be a periodic function with period k. Assume that $\sum_{1\leq n\leq k} \chi(n) = 0$. Show that then

$$L(\chi, s) := \sum_{n=1}^{\infty} \chi(n)n^{-s}$$

satisfies

$$L(\chi, s) + s\zeta(s+1)k^{-s-1}\sum_{n=1}^{k}\chi(n) =$$
$$\sum_{n=1}^{k}\chi(n)n^{-s} + \sum_{m=1}^{\infty}\sum_{n=1}^{k}\chi(n)[(mk+n)^{-s} - (mk)^{-s} + sn(mk)^{-s-1}],$$

where the last sum over m and k converges absolutely and uniformely on compact subsets of the half plane $\mathrm{Res}(s) > -1$. Thus this formula provides an analytic continuation of $L(\chi, s)$ to that half plane.

b) Under the same assumptions as in Part a) show that

$$L(\chi, 0) = \frac{-1}{k}\sum_{n=1}^{k} n\chi(n).$$

c) Using the functional equation of $L(\chi_D, s)$ prove that (1) leads to the simpler formula

$$L(\chi_D, 0) = \frac{2h}{\omega}.$$

Moreover, conclude from Part b) that

$$L(\chi_D, 0) = \frac{-1}{|D|} \sum_{n=1}^{|D|} n\chi_D(n).$$

Literatur

[1] Stark, H. M., *Dirichlet's Class-Number Formula Revisited*, Contemporary Mathematics Volume 143 1993, pp 571-577.