Serie 1

1. Consider the action of $\mathrm{SL}_2(\mathbb{R})$ on the set $\mathrm{Mat}_2(\mathbb{R})$ of 2×2-matrices with coefficients in $\mathbb{R}$ defined by $\gamma \circ M := M[\gamma^{-1}] := (\gamma^{-1})^t M \gamma^{-1}$, where $M \in \mathrm{Mat}_2(\mathbb{R})$ and $\gamma \in \mathrm{SL}_2(\mathbb{R})$.

a) Show that this action restricts to the subset $\mathcal{SP}_2(\mathbb{R}) \subset \mathrm{Mat}_2(\mathbb{R})$ of positive definite symmetric quadratic matrices with determinant 1.

Let us moreover associate to any element $z = x + iy$ in the upper half plane $\mathbb{H}$ the matrix

$$M_z := \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \left[\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \right] = \frac{1}{y} \begin{pmatrix} 1 & -x \\ -x & x^2 + y^2 \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{R}).$$

b) Show that the association $z \mapsto M_z$ defines an $\mathrm{SL}_2(\mathbb{R})$-equivariant bijection

$$\phi : \mathbb{H} \to \mathcal{SP}_2(\mathbb{R}).$$

2. Let D be any negative integer that is 0 or 1 modulo 4. Let $\mathcal{Q}_D$ be the set of quadratic forms $[A, B, C] := Ax^2 + Bxy + Cy^2 \in \mathbb{Z}[x, y]$ such that $A > 0$ and $B^2 - 4AC = D$ and such that the greatest common divisor of A, B, C is 1. This is called the set of *positive definite primitive quadratic forms of discriminant* D.

For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and any $Q \in \mathcal{Q}_D$ we set $(\gamma Q)[x, y] := Q[ax + by, cx + dy] \in \mathbb{Z}[x, y]$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma^{-1}$.

a) Show that this defines an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{Q}_D$ and show that the association

$$[A, B, C] \mapsto \psi([A, B, C]) := \frac{2}{\sqrt{|D|}} \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix}$$

defines an $\mathrm{SL}_2(\mathbb{Z})$-equivariant map $\psi : \mathcal{Q}_D \to \mathcal{SP}_2(\mathbb{R})$.

The orbits of this action are called *equivalence classes* of $\mathcal{Q}_D$.

b) Show that $\phi^{-1} \circ \psi$ sends any quadratic form $[A, B, C] \in \mathcal{Q}_D$ to its unique root $\frac{-B + i\sqrt{|D|}}{2A}$ in $\mathbb{H}$.

c) Show that any equivalence class of $\mathcal{Q}_D$ has a unique representative in the set

$$\mathcal{Q}_D^{\mathrm{red}} := \{[A, B, C] \in \mathcal{Q}_D | \ -A < B \leq A < C \ \textit{or} \ 0 \leq B \leq A = C\}$$

of *reduced quadratic forms* of $\mathcal{Q}_D$.

d) Conclude that the set of equivalence classes of $\mathcal{Q}_D$ is finite. Its order $h(D) := |\operatorname{SL}_2(\mathbb{Z})\backslash\mathcal{Q}_D|$ is called the *class number* of D.

3. Let $\tau = x + iy \in \mathbb{H}$, $q := e^{2\pi i\tau}$, $\sigma_1(n) := \sum_{d|n} d$. We define **Eisenstein series** of weight 2:

$$G_2(\tau) := \frac{1}{2}\sum_{0\neq n\in\mathbb{Z}}\frac{1}{n^2} + \sum_{0\neq m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^2}$$

$$G_2^*(\tau) := G_2(\tau) - \frac{\pi}{2y}$$

$$G_{2,\varepsilon}(\tau) := \frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m\tau+n)^2}\frac{1}{|m\tau+n|^{2\varepsilon}}, \text{ for } \varepsilon > 0$$

a) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Check that $G_{2,\varepsilon}$ converges absolutely and locally uniformely and satisfies: $G_{2,\varepsilon}(\gamma\tau) = (c\tau+d)^2|c\tau+d|^{2\varepsilon}G_{2,\varepsilon}(\tau)$.

b) For $\varepsilon > -\frac{1}{2}, \tau \in \mathbb{H}$ let:

$$I_\varepsilon(\tau) := \int_{-\infty}^{\infty}\frac{dt}{(\tau+t)^2|\tau+t|^{2\varepsilon}} \quad \text{and} \quad I(\varepsilon) := \int_{-\infty}^{\infty}(t+i)^{-2}(t^2+1)^{-\varepsilon}dt$$

Consider $G_{2,\varepsilon}(\tau) - \sum_{m=1}^{\infty} I_e(m\tau)$. Use the mean-value theorem to show that it converges absolutely and locally uniformly for $\varepsilon > -\frac{1}{2}$ and that its limit as $\varepsilon \to 0$ is $G_2(\tau)$.

c) Show that: $I_\varepsilon(x+iy) = \frac{I(\varepsilon)}{y^{1+2\varepsilon}}$ and $I'(0) = -\pi$.
Use this to show that: $\lim_{\varepsilon\to\infty} G_{2,\varepsilon}(\tau) = G_2^*(\tau)$.
Hence G_2^* transforms like a modular form of weight 2.

d) Conclude that:

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2G_2(z) - \pi ic(cz+d).$$

4. Recall that Möbius transformations form the group of automorphisms of the Riemann sphere, and that $\operatorname{Aut}(\hat{\mathbb{C}}) \cong \operatorname{PSL}(2,\mathbb{C})$.

a) Show that any non-trivial automorphism $A \in \operatorname{Aut}(\hat{\mathbb{C}})$, $A \neq 1$, has at least one and at most two fixed points.

b) If $A \in \operatorname{Aut}(\hat{\mathbb{C}})$ has two distinct fixed points z_-, $z_+ \in \hat{\mathbb{C}}$, then show that A is conjugate to the LFT $z \mapsto \mu\cdot z$, for some $\mu \in \mathbb{C}^\times$ (called the multiplier).

c) Show that : If $A \in \operatorname{Aut}(\hat{\mathbb{C}})$ has exactly one fixed point, then it is conjugate to the translation $z \mapsto z+1$.

A non-trivial $A \in \operatorname{Aut}(\hat{\mathbb{C}})$ is called

- parabolic iff A has exactly one fixed point,
- elliptic iff $|\mu| = 1$
- hyperbolic iff $\mu \in \mathbb{R}_{>0}$,
- loxodromic otherwise.

d) Let $z \in \hat{\mathbb{C}}$. Describe (or sketch) the orbits $\{A^n z : n \in \mathbb{Z}\}$ on the sphere for each type of motion .

e) One can also classify the motions algebraically. Check that the trace is not well-defined on $\mathrm{PSL}(2, \mathbb{C})$ but that its square is. Then give a characterization of parabolic, elliptic, hyperbolic and loxodromic motions using the square of the trace. (Note that the trace is conjugation-invariant.)

Remark : The loxodromic case does not appear for $\mathrm{PSL}(2, \mathbb{R})$.

5. Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be the generators of the full modular group $SL_2(\mathbb{Z})$ and let p be a prime. For $0 \leq l < p$ we set $\alpha_l := ST^l$ and $\alpha_p := 1$.

a) Show that: $SL_2(\mathbb{Z}) = \bigcup_{l=0}^{p} \alpha_l^{-1} \Gamma_0(p) = \bigcup_{l=0}^{p} \Gamma_0(p) \alpha_l$.

b) Let $\mathcal{F} = SL_2(\mathbb{Z}) \backslash \mathbb{H}$ denote the usual fundamental domain of $SL_2(\mathbb{Z})$ and set $\mathcal{F}_p := \bigcup_{l=0}^{p} \alpha_l \mathcal{F}$. Show that $\mathcal{F}_p$ is a fundamental domain of $\Gamma_0(p)$.

c) Draw a picture of $\mathcal{F}_2$. What are the cusps of $\Gamma_0(2)$?