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Serie 3

1. **a)** Show that $j : \Gamma\backslash\overline{\mathbb{H}} \to \mathbb{C} \cup \{\infty\}$ gives a bijection between $\Gamma\backslash\overline{\mathbb{H}}$ and the Riemann sphere $\mathbb{C} \cup \{\infty\}$. More precisely, j maps ∞ to ∞ and induces a bijection between $\Gamma\backslash\mathbb{H}$ and the complex plane $\mathbb{C}$.

b) Let $\mathcal{F}$ be the standard fundamental domain for the action of Γ on $\overline{\mathbb{H}}$.

Find $j(i)$, $j(\varrho)$ and determine all $\tau \in \mathcal{F}$ such that $j(\tau) \in \mathbb{R}$. Then show that j maps the left half of $\mathcal{F}$ onto $\mathbb{H}$ and the right half of $\mathcal{F}$ onto the lower half plane.

2. Given a lattice $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$, let

$$g_2 := g_2(L) = g_2(\omega_1, \omega_2) \;=\; 60 \sum_{m,n} (m\omega_1 + n\omega_2)^{-4}\,,$$
$$g_3 := g_3(L) = g_3(\omega_1, \omega_2) \;=\; 140 \sum_{m,n} (m\omega_1 + n\omega_2)^{-6}\,.$$

These two functions g_2 and g_3 are called the invariants of L. Observe that $g_2^3 - 27g_3^2 \neq 0$.

Prove that given two complex numbers a_2 and a_3 satisfying $a_2^3 - 27a_3^2 \neq 0$, there exist complex numbers ω_1 and ω_2 such that ω_1/ω_2 is not real, and $g_2(\omega_1, \omega_2) = a_2$, $g_3(\omega_1, \omega_2) = a_3$.

3. Let $V = \{(z, w) \in \mathbb{C}^2 : z^3 - 27w^2 = 0\}$. Let S^3 denote the 3-sphere. Then $T = V \cap S^3$ is the trefoil knot.

Prove that the space of lattices $\mathrm{SL}(2, \mathbb{Z})\backslash\mathrm{SL}(2, \mathbb{R})$ can be identified with the complement of the trefoil knot $S^3\backslash T$.

Note : In fact, they are even diffeomorphic.

4. Prove Picard's Theorem :

Every non-constant entire function attains every complex value with at most one exception.