Solutions 5

1. Let $a := (a(n))_{n\geq 1}$ be a sequence of complex numbers. We say that the sequence a is multiplicative if $a(mn) = a(m)a(n)$ for all coprime integers m, n (i.e. $\gcd(m,n) = 1$ for all $m, n \geq 1$). The sequence a is called completely multiplicative if $a(mn) = a(m)a(n)$ holds in general.

Let $\sigma_a \in \mathbb{R}$ be such that

$$L(s) := \sum_{n\geq 1} \frac{a(n)}{n^s}$$

converges absolutely on the half plane of convergence $H(a) := \{s \in \mathbb{C} | \ \mathrm{Re}(s) > \sigma_a\}$.

a) Show that if a is multiplicative, then

$$L(s) = \prod_p \left(\sum_{k\geq 0} \frac{a(p^k)}{p^{ks}} \right)$$

for all $s \in H(a)$.

b) Show that if a is completely multiplicative, then

$$L(s) = \prod_p \frac{1}{1 - a(p)p^{-s}}$$

for all $s \in H(a)$.

2. Let $f\ \mathbb{R}_+^\times \to \mathbb{C}$ be a continuous function such that $f(y)y^{s-1} \in L^1(\mathbb{R}_+^\times)$ for each

$$s \in \langle \alpha, \beta \rangle := \{s \in \mathbb{C} \ | \ \alpha < \mathrm{Re}(s) < \beta\}$$

the fundamental strip determined by $\alpha < \beta \in \mathbb{R} \cup \infty$. Its Mellin transform is defined by

$$\mathcal{M}(f)(s) := \int_0^\infty f(y)y^s \frac{dy}{y}$$

for all $s \in \langle \alpha, \beta \rangle$.

a) Show that $\mathcal{M}(f)$ is well-defined and holomorphic.

b) Prove the following identities for $\mathcal{M}(f)$:

$$\begin{aligned}
\mathcal{M}(y^\nu f(y))(s) &= \mathcal{M}(f(y))(s+\nu) \\
\mathcal{M}(f(\nu y))(s) &= \nu^{-s}\mathcal{M}(f(y))(s) \\
\mathcal{M}(f(y^\nu))(s) &= \frac{1}{\nu}\mathcal{M}(f(y))\left(\frac{s}{\nu}\right) \\
\mathcal{M}\left(\frac{1}{y}f\left(\frac{1}{y}\right)\right)(s) &= \mathcal{M}(f(y))(1-s) \\
\frac{d}{ds}\mathcal{M}(f(y))(s) &= \mathcal{M}(f(y)\log y)(s) \\
\mathcal{M}\left(\frac{d}{dy}f(y)\right)(s) &= -(s-1)\mathcal{M}(f(y))(s-1)
\end{aligned}$$

where $\nu > 0$.

3. Recall the Gamma function $\Gamma(s) = \int_0^\infty e^{-t}t^s\frac{dt}{t}$ defined for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. Prove that

a) The function $\Gamma(s)$ can be analytically continued to the whole complex plane into a meromorphic function whose poles are exactly non-positive integers and satisfies the functional equation $\Gamma(s+1) = \Gamma(s)$.

b) Show that the meromorphic continuation satisfies $\Gamma(s) = \sum_{k=0}^\infty \frac{(-1)^k}{k!(k+s)} + \int_1^\infty e^{-y}y^s\frac{dy}{y}$ and conclude that $\mathrm{Res}(\Gamma, -n) = \frac{(-1)^n}{n!}$.

c) Prove the reflection formula $\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}$ and conclude that $\frac{1}{\Gamma(s)}$ is an entire function of s.

d) Compute $\Gamma\left(\frac{1}{2}\right)$ and prove the duplication formula $\Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = 2^{\frac{1}{2}-2s}\sqrt{2\pi}\,\Gamma(2s)$.

e) Show that

$$\mathcal{M}\left(e^{-y^2}\right)(s) = \frac{1}{2}\Gamma\left(\frac{s}{2}\right) \qquad \text{for any } s \in H(0),$$

$$\mathcal{M}\left(\frac{e^{-y}}{1-e^{-y}}\right)(s) = \Gamma(s)\zeta(s) \qquad \text{for any } s \in H(1).$$

4. **a)** Take a modular form $f \in \mathcal{M}_k(\Gamma)$ with q-expansion $f = \sum a(n)q^n$. Let χ be a character mod p, where p is a prime, and set

$$f_\chi(z) = \sum a(n)\chi(n)q^n.$$

Show that $f_\chi \in \mathcal{M}_k(\Gamma_0(p^2), \chi^2)$, i.e.

$$f_\chi(\gamma z) = \chi(d)^2(cz+d)^k f(z).$$

Moreover, show that if $f \in \mathcal{S}_k(\Gamma)$, then $f_\chi \in \mathcal{S}_k(\Gamma_0(p^2), \chi^2)$.

Siehe nächstes Blatt!

b) Given N a positive integer, let $\omega_N := \begin{pmatrix} & -1 \\ N & \end{pmatrix}$. Show that ω_N normalizes $\Gamma_0(N)$ and that if $f \in \mathcal{M}_k(\Gamma_0(N))$, then

$$f|_{\omega_N} \;=\; N^{-k/2} z^{-k/2} f\left(\frac{-1}{Nz}\right)$$

is also in $\mathcal{M}_k(\Gamma_0(N))$.

c) Let $f \in \mathcal{S}_k(\Gamma)$, and let χ be a character mod p. Show that $f_\chi|_{\omega_{p^2}} = \frac{\tau(\chi)^2}{p} f_{\overline{\chi}}$, where $\tau(\chi) = G(1, \chi)$ denotes the Gauss sum.

5. Let again $f \in \mathcal{S}_k(\Gamma)$, and let χ be a Dirichlet character mod p. Set

$$L(f, \chi, s) \;=\; \sum_{n \geq 1} \frac{a(n)\chi(n)}{n^s} \qquad \text{and} \qquad \Lambda(f, \chi, s) \;=\; \left(\frac{p}{2\pi}\right)^s \Gamma(s) L(f, \chi, s).$$

Prove the functional equation $\Lambda(f, \chi, s) = i^k \frac{\tau(\chi)^2}{p} \Lambda(f, k - s, \overline{\chi})$.