Serie 7

1. Let $z \in \mathbb{H}$ and consider the Θ-function defined by

$$\Theta_z(t) \;=\; \sum_{m,n\in\mathbb{Z}} e^{-\pi t \frac{|mz+n|^2}{y}}$$

for all $t > 0$.

a) Show that Θ_z satisfies the functional equation $\Theta_z(t) \;=\; \frac{1}{t}\Theta_z\left(\frac{1}{t}\right)$.

For all $s \in \langle 1, \infty\rangle$, let

$$E(z,s) \;=\; \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s \;=\; \sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} \frac{y^s}{|cz+d|^{2s}}$$

and

$$E^*(z,s) \;=\; \pi^{-s}\Gamma(s)2\zeta(2s)E(z,s) \;=\; \pi^{-s}\Gamma(s)\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{y^s}{|mz+n|^{2s}}.$$

b) Check that $E(\gamma z, s) = E(z,s)$ for all $\gamma \in \Gamma$ and show that

$$E^*(z,s) \;=\; \int_0^\infty \left(\Theta_z(t) - 1\right) t^s \frac{dt}{t}.$$

c) Show that $E^*(z,s)$ has a meromorphic continuation to the whole complex s-plane with single poles at $s = 0$ and $s = 1$ with residues -1 and 1 respectively. Finally, prove the functional equation $E^*(z,1-s) = E^*(z,s)$.

2. Let $\varphi : \mathbb{H} \to \mathbb{C}$ be an analytic function such that $\varphi(\gamma z) = \varphi(z)$ for all $\gamma \in \Gamma$ and $\varphi(z) = O\left(y^{-C}\right)$ as $y \to \infty$ for all $C > 0$. Such a function has a Fourier expansion of the form $\varphi(z) = \sum_{n\in\mathbb{Z}} \varphi_n(y) e^{2\pi i n x}$ where $\varphi_n(y) = \int_0^1 \varphi(x+iy)e^{-2\pi i n x}dx$. Set

$$\Lambda_\varphi(s) \;=\; \pi^{-s}\Gamma(s)2\zeta(2s)\mathcal{M}(\varphi_0)(s-1)$$

for all $s \in \langle 1, \infty\rangle$.

a) Show that $\mathcal{M}(\varphi_0)(s)$ is indeed well-defined on the fundamental strip $\langle 0, \infty\rangle$ and that it is bounded in every vertical strip strictly contained in $\langle 0, \infty\rangle$.

b) Check that Λ_φ has the following integral representation

$$\Lambda_\varphi(s) \;=\; \langle \varphi, \overline{E^*(\cdot, s)} \rangle \;=\; \int_{\mathcal{F}} \varphi(z) E^*(z,s) d\mu(z)$$

where $\mathcal{F}$ denotes a fundamental domain for Γ.

c) Prove that Λ_φ has a meromorphic continuation to the whole complex plane with simple poles at $s = 0$ and $s = 1$ with residues $\mp \int_{\mathcal{F}} \varphi(z) d\mu(z)$. It is bounded in any vertical strip (that does not contain a pole) and satisfies the functional equation

$$\Lambda_\varphi(s) \;=\; \Lambda_\varphi(1-s).$$

N.B. This is the simplest case of the Rankin–Selberg method.

Let $f = \sum a_n q^n \in \mathcal{S}_k(\Gamma)$ and $g = \sum b_n q^n \in \mathcal{M}_k(\Gamma)$ and set $\phi = f\overline{g}y^k$. We define

$$\begin{aligned} L(f \times g, s) &= 2\zeta(2s - 2k + 2) \sum_{n \geq 1} a_n \overline{b_n} n^{-s}, \\ \Lambda(f \times g, s) &= \pi^{k-1}(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+1)L(f \times g, s). \end{aligned}$$

For simplicity, we will assume that $b_n = \overline{b_n}$ for all n.

The L-series $L(f \times g, s)$ is called the Rankin–Selberg convolution of f and g.

d) Check that ϕ satisfies the same properties as the function φ at the beginning of the exercise. Show that for all $s \in \langle 0, \infty \rangle$

$$\mathcal{M}(\phi_0)(s) \;=\; (4\pi)^{-(s+k)}\Gamma(s+k) \sum_{n \geq 1} a_n \overline{b_n} n^{-(s+k)}.$$

e) Prove that $\Lambda(f \times g, s)$ has a meromorphic continuation to the whole complex plane with simple poles at $s = k$ and $s = k-1$ with residues $\pm \langle f, g \rangle$. It is bounded in any vertical strip (that does not contain a pole) and satisfies the functional equation

$$\Lambda(f \times g, s) \;=\; \Lambda(f \times g, 2k - 1 - s).$$

Hint: Show first that $\Lambda_\phi(s) = \Lambda(f \times g, s + k - 1)$.

3. The MacDonald–Bessel function is given by

$$K_s(y) \;=\; \frac{1}{2} \int_0^\infty e^{-\frac{y}{2}\left(t + \frac{1}{t}\right)} t^s \frac{dt}{t}$$

for all $y > 0$, $s \in \mathbb{C}$. It is entire as a function in s and decays rapidly as $y \to \infty$. Moreover, one can show by a change of variable that $K_s(y) = K_{-s}(y)$.

a) Set

$$I_s(a) \;=\; \int_{\mathbb{R}} \frac{e^{iau}}{(u^2+1)^s}du$$

for all $a \in \mathbb{R}$, $s \in \langle 1/2, \infty\rangle$. Prove that

$$\Gamma(s)I_s(a) \;=\; \begin{cases} \sqrt{\pi}\,\Gamma(s-1/2) & a = 0 \\ 2\sqrt{\pi}\,\left|\frac{a}{2}\right|^{s-1/2} K_{s-1/2}(|a|) & a \neq 0. \end{cases}$$

Let $s \in \langle 1, \infty\rangle$ and consider the Fourier expansion $E^*(z,s) = \sum_{n\in\mathbb{Z}} a_n(y,s)e^{2\pi i n x}$ with coefficients

$$\begin{aligned} a_0(y,s) &= 2\Lambda(2s)y^s + 2\Lambda(2s-1)y^{1-s} \\ a_n(y,s) &= 4\sqrt{y}\,|n|^{s-1/2}\sigma_{1-2s}(|n|)K_{s-1/2}(2\pi|n|y) \end{aligned}$$

where $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

b) Prove that each coefficient $a_n(y,s)$, $n \neq 0$, has an analytical continuation to an entire function and satisfies the functional equation

$$a_n(y,s) \;=\; a_n(y,1-s).$$

c) Show that $\Lambda(s)$ has a meromorphic continuation to the whole complex plane with simple poles at $s = 0, 1$ with residues ∓ 1, and that it satisfies the functional equation

$$\Lambda(s) \;=\; \Lambda(1-s).$$

4. **a)** Show that

$$\mathcal{M}(K_s)(w) \;=\; 2^{w-2}\Gamma\left(\frac{w+s}{2}\right)\Gamma\left(\frac{w-s}{2}\right).$$

b) Show that

$$\mathcal{M}\left((E^*(iy,s) - a_0(y,s))\right)(w) \;=\; 2\Lambda(w+s)\Lambda(s-w).$$

Hint: Show first that $\sum_{n\geq 1}\sigma_w(n)n^{-s} = \zeta(s)\zeta(s-w)$.