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Solutions 1

1. Consider the action of $\mathrm{SL}_2(\mathbb{R})$ on the set $\mathrm{Mat}_2(\mathbb{R})$ of 2×2-matrices with coefficients in $\mathbb{R}$ defined by $\gamma \circ M := M[\gamma^{-1}] := (\gamma^{-1})^t M \gamma^{-1}$, where $M \in \mathrm{Mat}_2(\mathbb{R})$ and $\gamma \in \mathrm{SL}_2(\mathbb{R})$.

a) Show that this action restricts to the subset $\mathcal{SP}_2(\mathbb{R}) \subset \mathrm{Mat}_2(\mathbb{R})$ of positive definite symmetric quadratic matrices with determinant 1.

Solution : This follows immediately from the definitions, from basic properties of transposing matrices and from the multiplicativity of the determinant.

Let us moreover associate to any element $z = x + iy$ in the upper half plane $\mathbb{H}$ the matrix

$$M_z := \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \left[\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \right] = \frac{1}{y} \begin{pmatrix} 1 & -x \\ -x & x^2 + y^2 \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{R}).$$

b) Show that the association $z \mapsto M_z$ defines an $\mathrm{SL}_2(\mathbb{R})$-equivariant bijection

$$\phi : \mathbb{H} \to \mathcal{SP}_2(\mathbb{R}).$$

Solution : By construction, M_z is symmetric with determinant 1 and has positive principal minors, i.e. $M_z \in \mathcal{SP}_2(\mathbb{R})$, for any $z \in \mathbb{H}$. The induced map $\phi : \mathbb{H} \to \mathcal{SP}_2(\mathbb{R})$ is injective as can readily be read off its construction. Conversely, let $M \in \mathcal{SP}_2(\mathbb{R})$. As M has positive first principal minor and as it is symmetric, there exist $x, y \in \mathbb{R}$, where $y \neq 0$, such that $M = \frac{1}{y} \begin{pmatrix} 1 & -x \\ -x & \lambda \end{pmatrix}$ for some $\lambda \in \mathbb{R}$. As the determinant of M is 1, we get $\lambda = x^2 + y^2$ and hence

$$M = \frac{1}{y} \begin{pmatrix} 1 & -x \\ -x & x^2 + y^2 \end{pmatrix}.$$

This shows that ϕ is surjective.

In order to see that ϕ is $\mathrm{SL}_2(\mathbb{R})$-equivariant, we consider any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and any $M = \frac{1}{y} \begin{pmatrix} 1 & -x \\ -x & x^2 + y^2 \end{pmatrix} \in \mathcal{SP}_2(\mathbb{R})$. We have $\gamma^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and compute

$$\begin{aligned} \gamma \circ M = M[\gamma^{-1}] &= \frac{1}{y} \begin{pmatrix} (cx+d)^2 + (cy)^2 & -[(ax+b)(cx+d) + acy^2] \\ * & * \end{pmatrix} \\ &= \frac{(cx+d)^2 + (cy)^2}{y} \begin{pmatrix} 1 & \frac{-[(ax+b)(cx+d)+acy^2]}{(cx+d)^2+(cy)^2} \\ * & * \end{pmatrix} \end{aligned}$$

and

$$\gamma(x+iy) = \frac{a(x+iy)b}{c(x+iy)+d} = \frac{(ax+b)(cx+d)+acy^2+iy}{(cx+d)^2+(ay)^2},$$

where for the imaginary part we have used that $ad-bc=1$. From this we see that indeed $\phi(\gamma z) = \gamma \circ \phi(z)$.

2. Let D be any negative integer that is 0 or 1 modulo 4. Let $\mathcal{Q}_D$ be the set of quadratic forms $[A,B,C] := Ax^2+Bxy+Cy^2 \in \mathbb{Z}[x,y]$ such that $A>0$ and $B^2-4AC=D$ and such that the greatest common divisor of A,B,C is 1. This is called the set of *positive definite primitive quadratic forms of discriminant* D.

For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and any $Q \in \mathcal{Q}_D$ we set $(\gamma Q)[x,y] := Q[ax+by, cx+dy] \in \mathbb{Z}[x,y]$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma^{-1}$.

a) Show that this defines an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{Q}_D$ and show that the association

$$[A,B,C] \mapsto \psi([A,B,C]) := \frac{2}{\sqrt{|D|}} \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix}$$

defines an $\mathrm{SL}_2(\mathbb{Z})$-equivariant map $\psi : \mathcal{Q}_D \to \mathcal{SP}_2(\mathbb{R})$.

Solution : Consider any $Q := [A,B,C] \in \mathcal{Q}_D$. By construction $\psi(Q)$ is a symmetric matrix with determinant 1. As moreover $\frac{2A}{\sqrt{|D|}} > 0$ is the first principal minor of $\psi(Q)$, we see that $\psi(Q)$ is positive definite. Thus $\psi(Q) \in \mathcal{SP}_2(\mathbb{R})$.

We claim that for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have $\psi(\gamma Q) = \gamma\psi(Q)$. As ψ is injective, this will show both that $(\gamma, Q) \mapsto \gamma Q$ defines an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{Q}_D$ and that ψ is $\mathrm{SL}_2(\mathbb{Z})$-equivariant. In order to prove the claim it is enough to consider the cases

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

because these two matrices generate $\mathrm{SL}_2(\mathbb{Z})$. The claim for any of these two matrices however is immediately checked via a straightforward computation.

The orbits of this action are called *equivalence classes* of $\mathcal{Q}_D$.

b) Show that $\phi^{-1} \circ \psi$ sends any quadratic form $[A,B,C] \in \mathcal{Q}_D$ to its unique root $\frac{-B+i\sqrt{|D|}}{2A}$ in $\mathbb{H}$.

Solution : We have

$$\psi([A,B,C]) = \frac{2}{\sqrt{|D|}} \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} = \frac{2A}{\sqrt{|D|}} \begin{pmatrix} 1 & \frac{B}{2A} \\ \frac{B}{2A} & \frac{C}{A} \end{pmatrix}$$

for any $[A,B,C] \in \mathcal{Q}_D$ and therefore $\phi^{-1} \circ \psi([A,B,C]) = \frac{-B+i\sqrt{|D|}}{2A}$.

c) Show that any equivalence class of $\mathcal{Q}_D$ has a unique representative in the set

$$\mathcal{Q}_D^{\text{red}} := \{[A,B,C] \in \mathcal{Q}_D| \; -A < B \le A < C \textit{ or } 0 \le B \le A = C\}$$

of *reduced quadratic forms* of $\mathcal{Q}_D$.

Solution : By a theorem of the lecture, any element in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ has a unique representative in the strict fundamental domain $\mathcal{F} \subset \mathbb{H}$. In particular, as $\psi \circ \phi^{-1}$ is $\mathrm{SL}_2(\mathbb{Z})$-equivariant, any equivalence class of $\mathcal{Q}_D$ has a unique representative $[A,B,C]$ such that $\frac{-B+i\sqrt{|D|}}{2A} \in \mathcal{F}$. We claim that such representatives are precisely the reduced quadratic forms. By definition of $\mathcal{F}$, we have $\frac{-B+i\sqrt{|D|}}{2A} \in \mathcal{F}$ if and only if $\frac{-1}{2} \le \frac{-B}{2A} < \frac{1}{2}$, i.e. $-A < B \le A$, and

$$\frac{\sqrt{|D|}}{4A^2} > \sqrt{1 - \frac{B^2}{4A^2}} \quad \text{if} \quad \frac{-B}{2A} > 0$$

and

$$\frac{\sqrt{|D|}}{4A^2} \ge \sqrt{1 - \frac{B^2}{4A^2}} \quad \text{if} \quad \frac{-B}{2A} \le 0.$$

Using $B^2 - 4AC = D$, we get that $\frac{\sqrt{|D|}}{4A^2} > \sqrt{1 - \frac{B^2}{4A^2}}$ respectively $\frac{\sqrt{|D|}}{4A^2} \ge \sqrt{1 - \frac{B^2}{4A^2}}$ if and only if $A < C$ respectively $A \le C$. Therefore $\frac{-B+i\sqrt{|D|}}{2A} \in \mathcal{F}$ if and only if $-A < B \le A < C$ or $0 \le B \le A = C$. These are precisely the properties defining $\mathcal{Q}_D^{\text{red}}$.

d) Conclude that the set of equivalence classes of $\mathcal{Q}_D$ is finite. Its order $h(D) := |\,\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{Q}_D|$ is called the *class number* of D.

Solution : Consider any $[A,B,C] \in \mathcal{Q}_D^{\text{red}}$. As $C \ge A \ge |B|$, we have that $|D| = 4AC - B^2 \ge 3A^2$ and thus

$$A, |B| \le \sqrt{\frac{|D|}{3}}.$$

As $C = \frac{B^2-D}{4A}$ is uniquely determined by A, B and D, this implies that $\mathcal{Q}_D^{\text{red}}$ is a finite set. By Part c) we moreover have that $|\mathcal{Q}_D^{\text{red}}| = |\,\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{Q}_D|$ which finishes the Exercise.

3. Let $\tau = x + iy \in \mathbb{H}$, $q := e^{2\pi i\tau}$. We define **Eisenstein series** of weight 2:

$$G_2(\tau) := \frac{1}{2}\sum_{0 \ne n \in \mathbb{Z}} \frac{1}{n^2} + \sum_{0 \ne m \in \mathbb{Z}}\sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^2},$$

$$G_2^*(\tau) := G_2(\tau) - \frac{\pi}{2y}$$

$$G_{2,\varepsilon}(\tau) := \frac{1}{2}\sum_{m,n \in \mathbb{Z}}{}' \frac{1}{(m\tau+n)^2}\frac{1}{|m\tau+n|^{2\varepsilon}}, \text{ for } \varepsilon > 0$$

a) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Check that $G_{2,\varepsilon}$ converges absolutely and locally uniformely and satisfies: $G_{2,\varepsilon}(\gamma\tau) = (c\tau+d)^2|c\tau+d|^{2\varepsilon}G_{2,\varepsilon}(\tau)$.

Proof:
Let $\tau \in \mathbb{H}$, $\varepsilon > 0$, $2 < k \in \mathbb{R}$ and let $\Lambda_\tau = \tau\mathbb{Z} + \mathbb{Z}$. Then the number of pairs $(m,n) \in \mathbb{Z}^2$ with $N \leq |m\tau + n| < N+1$ is the number of lattice points in the annulus of area $\pi(N+1)^2 - \pi N^2$, so it is $O(N)$ and the series $\sum'_{m,n\in\mathbb{Z}} \frac{1}{|m\tau+n|^k}$ is majorized by the sum $\sum_{N=1}^{\infty} N^{1-k}$ which converges absolutely for $k > 1$. Since $2+2\varepsilon > 2$ we get that $G_{2,\varepsilon}$ converges absolutely and locally uniformly (so $G_{2,\varepsilon}$ is holomorphic). We use that the matrix vector multiplication from the right $(m', n') = (ma + nc, mb + nd) = (m,n)\gamma$ with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ gives a bijection from $\mathbb{Z}^2\backslash(0,0)$ to itself. Furthermore we have:

$$m\gamma\tau + n = \frac{m(a\tau+b) + n(c\tau+d)}{j(\gamma,\tau)} = \frac{(ma+nc)\tau + (mb+nd)}{j(\gamma,\tau)} = \frac{m'\tau+n'}{j(\gamma,\tau)}$$

The two previous facts give:

$$\begin{aligned} G_{2,\varepsilon}(\gamma\tau) &= \sum_{m,n\in\mathbb{Z}}{}' (m\gamma\tau+n)^2|m\gamma\tau+n|^{2\varepsilon} \\ &= \sum_{m',n'\in\mathbb{Z}}{}' j(\gamma,\tau)^2(m'\tau+n')^2|j(\gamma,\tau)|^{2\varepsilon}|m'\tau+n'|^{2\varepsilon} \\ &= (cz+d)^2|cz+d|^{2\varepsilon}G_{2,\varepsilon}(\tau) \end{aligned}$$

b) For $\varepsilon > -\frac{1}{2}, \tau \in \mathbb{H}$ let:

$$I_\varepsilon(\tau) := \int_{-\infty}^{\infty} \frac{dt}{(\tau+t)^2|\tau+t|^{2\varepsilon}} \quad \text{and} \quad I(\varepsilon) := \int_{-\infty}^{\infty} (t+i)^{-2}(t^2+1)^{-\varepsilon}dt$$

Consider $G_{2,\varepsilon}(\tau) - \sum_{m=1}^{\infty} I_e(m\tau)$. Use the mean-value theorem to show that it converges absolutely and locally uniformly for $\varepsilon > -\frac{1}{2}$ and that its limit as $\varepsilon \to 0$ is $G_2(\tau)$.

Proof:
Set $f(t) := (m\tau+t)^{-2}|m\tau+t|^{-2\varepsilon}$. We use $2a)$ to change the order of summation and to interchange integration and summation.

$$\begin{aligned} \widetilde{G_{2,\varepsilon}}(\tau) &:= G_{2,\varepsilon}(\tau) - \sum_{m=1}^{\infty} I_e(m\tau) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\varepsilon}} + \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\left[\frac{1}{(m\tau+n)^2}\frac{1}{|m\tau+n|^{2\varepsilon}} - \int_n^{n+1}\frac{dt}{(m\tau+t)^2|m\tau+t|^{2\varepsilon}}\right] \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\varepsilon}} + \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\int_n^{n+1}(f(n)-f(t))dt \end{aligned}$$

By the mean-value theorem we know that for $n \leq t \leq n+1$:

$$|f(t) - f(n)| \leq max_{n\leq u\leq n+1}|f'(u)| = O(|m\tau+n|^{-3-2\varepsilon})$$

Hence the limit $\lim_{\varepsilon\to 0}\widetilde{G_{2,\varepsilon}}(\tau)$ *exists and can be obtained by putting* $\varepsilon = 0$ *in each term:*

$$\begin{aligned}\lim_{\varepsilon\to 0}\widetilde{G_{2,\varepsilon}}(\tau) &= \frac{1}{2}\sum_{0\neq n\in\mathbb{Z}}^{\infty}\frac{1}{n^2} + \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\left[\frac{1}{(m\tau+n)^2} + \frac{1}{m\tau+n+1} - \frac{1}{m\tau+n}\right] \\ &= \frac{1}{2}\sum_{0\neq n\in\mathbb{Z}}^{\infty}\frac{1}{n^2} + \sum_{m=1}^{\infty}\left(\sum_{n=-\infty}^{\infty}\frac{1}{(m\tau+n)^2} - \sum_{n=-\infty}^{\infty}\frac{1}{(m\tau+n+1)(m\tau+n)}\right)\end{aligned}$$

Note that the last inner sum is given by telescoping sum $H_x := \sum_{n=0}^{\infty}\frac{1}{(x+n+1)(x+n)} = \frac{1}{x}$*:*

$$\sum_{n=-\infty}^{\infty}\frac{1}{(m\tau+n+1)(m\tau+n)} = H_{m\tau} + H_{-m\tau} = 0$$

Hence $\lim_{\varepsilon\to 0}\widetilde{G_{2,\varepsilon}}(\tau) = \frac{1}{2}\sum_{0\neq n\in\mathbb{Z}}^{\infty}\frac{1}{n^2} + \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(m\tau+n)^2} = G_2(\tau)$.

c) Show that: $I_\varepsilon(x+iy) = \frac{I(\varepsilon)}{y^{1+2\varepsilon}}$ and $I'(0) = -\pi$.
Use this to show that: $\lim_{\varepsilon\to 0} G_{2,\varepsilon}(\tau) = G_2^*(\tau)$.
Hence G_2^* transforms like a modular form of weight 2.

Proof:

$$\begin{aligned}I_\varepsilon(x+iy) &= \int_{-\infty}^{\infty}\frac{dt}{((x+t)+iy)^2((x+t)^2+y^2)^\varepsilon} \overset{(t\mapsto t-x)}{=} \int_{-\infty}^{\infty}\frac{dt}{(t+iy)^2(t^2+y^2)^\varepsilon} \\ &\overset{(t\mapsto ty)}{=} \frac{1}{y^{2+2\varepsilon}}\int_{-\infty}^{\infty}\frac{ydt}{(t+i)^2(t^2+1)^\varepsilon} = \frac{I(\varepsilon)}{y^{1+2\varepsilon}} \\ I'(0) &= -\int_{-\infty}^{\infty}\frac{\log(t^2+1)}{(t+i)^2}dt = \left.\frac{\log(t^2+1)}{t+i}\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{2t}{(t+i)(t^2+1)}dt \\ &= -\int_{-\infty}^{\infty}\frac{1}{(t+i)^2} + \frac{1}{t^2+1}dt = 0 - \int_{-\infty}^{\infty}\frac{1}{t^2+1}dt \\ &= -\arctan(t)|_{-\infty}^{\infty} = -\pi\end{aligned}$$

Hence $\sum_{m=1}^{\infty} I_\varepsilon(m\tau) = \frac{I(\varepsilon)\zeta(1+2\varepsilon)}{y^{1+2\varepsilon}}$ *with* $\zeta(1+2\varepsilon) = \frac{1}{2\varepsilon} + O(1)$. *Hence in the limit* $\varepsilon\to 0$ *this product converges to* $\frac{I'(0)}{2y} = \frac{-\pi}{2y}$. *This means that*

$$\lim_{\varepsilon\to 0} G_{2,\varepsilon}(\tau) = \lim_{\varepsilon\to 0}\left(\widetilde{G_{2,\varepsilon}}(\tau) - \sum_{m=1}^{\infty} I_\varepsilon(m\tau)\right) = G_2(\tau) - \frac{\pi}{2y} = G_2^*(\tau)$$

The modularity of G_2^* *now follows from part* a).

d) Conclude that:

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 G_2(z) - \pi ic(cz+d).$$

Proof:
Since $z \mapsto G_2(z) - \frac{\pi}{2y}$ is modular of weight 2, it follows that

$$\begin{aligned} G_2(\gamma z) - (cz+d)^2 G_2(z) &= \frac{\pi}{2y(\gamma z)} - (cz+d)^2 \frac{\pi}{2y} \\ &= \frac{\pi}{2y}\left(|cz+d|^2 - (cz+d)^2\right) \\ &= -\pi i c(cz+d). \end{aligned}$$

4. Recall that Möbius transformations form the group of automorphisms of the Riemann sphere, and that $\mathrm{Aut}(\hat{\mathbb{C}}) \cong \mathrm{PSL}(2,\mathbb{C})$.

a) Show that any $A \in \mathrm{Aut}(\hat{\mathbb{C}})$, $A \neq 1$, has at least one and at most two fixed points.

Solution : The quadratic polynomial equation $Az - z = 0$ has two solutions (which must not necessarily be distinct).

b) If $A \in \mathrm{Aut}(\hat{\mathbb{C}})$ has two distinct fixed points z_-, $z_+ \in \hat{\mathbb{C}}$, then show that A is conjugate to the LFT $z \mapsto \mu \cdot z$, for some $\mu \in \mathbb{C}^\times$ (called the multiplier).

Solution : We can conjugate A to an element with fixed points 0 and ∞, e.g. take

$$B(z) = \frac{z - z_-}{z - z_+},$$

then $C := BAB^{-1}$ is such an automorphism. Now consider the element associated to C in $\mathrm{PSL}(2,\mathbb{C})$:

$$\left.\begin{matrix} C(0) = 0 \\ C(\infty) = \infty \end{matrix}\right\} \implies C = \begin{pmatrix} \sqrt{\mu} & \\ & 1/\sqrt{\mu} \end{pmatrix}.$$

c) Show that : If $A \in \mathrm{Aut}(\hat{\mathbb{C}})$ has exactly one fixed point, then it is conjugate to the translation $z \mapsto z+1$.

Solution :

The solutions to the quadratic equation $Az - z = 0$ are

$$z_\pm = \frac{a - d \pm \sqrt{\mathrm{tr}(A)^2 - 4}}{2c}.$$

Then : $z_+ = z_-$ iff $\mathrm{tr}(A)^2 = 4$. On the other hand, A can be conjugate to a matrix element C with fixed point at ∞, therefore of the form

$$\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \qquad (a, b \in \mathbb{C}^*).$$

Solving $\left(a + a^{-1}\right)^2 = 4$, yields $a = \pm 1$. Finally,

$$\begin{pmatrix} 1/\sqrt{b} & \\ & \sqrt{b} \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \sqrt{b} & \\ & 1/\sqrt{b} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}.$$

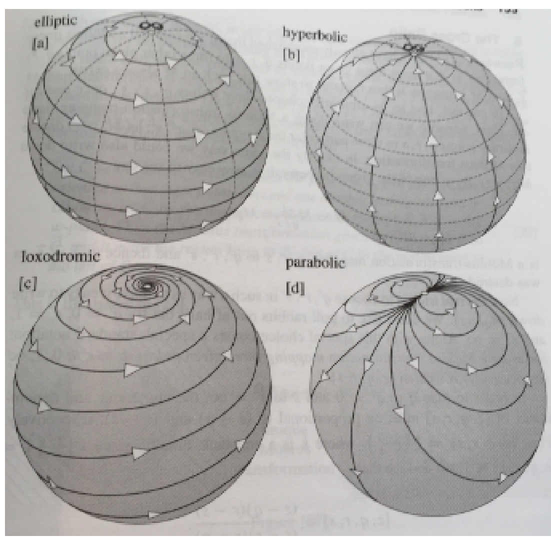
(d) Let $z \in \hat{\mathbb{C}}$. Describe (or sketch) the orbits $\{A^n z : n \in \mathbb{Z}\}$ on the sphere for each type of motion.

Solution : One way to visualise the orbits is via the stereographic projection. Let p be a point on the sphere that is neither the north pole nor the south pole. Under the stereographic projection, it is projected to a point $z \in \mathbb{C}^\times$. We consider successively the orbits generated by the transformations $A = A_{\text{ell}}, A_{\text{hyp}}, A_{\text{lox}}, A_{\text{par}}$ on $\mathbb{C}$.

By definition A_{ell} is of the form $A_{\text{ell}}(z) = e^{i\vartheta} z$ for some $\vartheta \in [0, 2\pi)$, that is, it rotates the plane around the origin. The pre images of $(A^n z)$ under the stereographic projection all lie at the same latitude. (See figure below.)

We now consider $A^n_{\text{hyp}}(n) = \mu^n z$ for $\mu \in \mathbb{R}_{>0}$. This transformation pushes z along the half-line starting at the origin and passing through z. Its pre images under the stereographic projection all lie on the same longitudinal line going from south pole to north pole. (In fact, one speaks of repulsive fixed point and attractive fixed point.) On the other hand, the orbit points $A^n_{\text{lox}}(z)$ are obtained by multiplying z with powers of a complex number μ (with $|\mu| \neq 1$). This is what accounts for the winding lines running from the repulsive fixed point at the south pole to the attractive fixed point at the north pole in the illustration [c] below.

Finally, for the parabolic case, consider $A_{\text{par}}(z) = z + 1$ and its orbit $\{\mathbb{Z} + i\text{Im}(z)\}$. The only fixed point is the point at ∞ that corresponds to the north pole on the sphere.



(e) One can also classify the motions algebraically. Check that the trace is not well-defined on $\mathrm{PSL}(2, \mathbb{C})$ but that its square is. Then give a characterization of parabolic, elliptic, hyperbolic and loxodromic motions using the square of the trace. (Note that the trace is conjugation-invariant.)

Remark : The loxodromic case does not appear for $\mathrm{PSL}(2, \mathbb{R})$.

Solution : The trace is clearly well-defined on $\mathrm{SL}(2, \mathbb{C})$ but not on $\mathrm{PSL}(2, \mathbb{C})$. Let $A \in \mathrm{SL}(2, \mathbb{C})$ be the preimage under the canonical projection $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{PSL}(2, \mathbb{C})$ of a non-parabolic transformation. Then, by (b), $\mathrm{tr}(A)^2 = \left(\sqrt{\mu} + \frac{1}{\sqrt{\mu}}\right)^2$. That is, if A

is elliptic, $\mathrm{tr}(A)^2 = \left(e^{i\vartheta/2} + e^{-i\vartheta/2}\right)^2 = 4\cos(\vartheta/2)^2$; if A is hyperbolic, $\mathrm{tr}(A)^2 = \left(e^{t/2} + e^{-t/2}\right)^2 = 4\cosh(t/2)^2$. Then A is

- parabolic iff $\mathrm{tr}(A)^2 = 4$,
- elliptic iff $\mathrm{tr}(A)^2 < 4$ and $\mathrm{tr}(A) \in \mathbb{R}$,
- hyperbolic iff $\mathrm{tr}(A)^2 > 4$ and $\mathrm{tr}(A) \in \mathbb{R}$,
- loxodromic otherwise.

5. Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be the generators of the full modular group $SL_2(\mathbb{Z})$ and let p be a prime. For $0 \le l < p$ we set $\alpha_l := ST^l$ and $\alpha_p := 1$.

a) Show that: $SL_2(\mathbb{Z}) = \bigcup_{l=0}^{p} \alpha_l^{-1}\Gamma_0(p) = \bigcup_{l=0}^{p} \Gamma_0(p)\alpha_l$.

Proof:
Let $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_2(\mathbb{Z})$, $\gamma \notin \Gamma_0(p)$. *We have to show that there exists* $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$ *such that* $\gamma = \beta\alpha_l$ *for* $0 \le l < p$*:*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \gamma \overset{!}{=} \beta\alpha_l = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & l \end{pmatrix} \text{ hence:}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{!}{=} \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} l & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} lA - B & A \\ lC - D & C \end{pmatrix}$$

We set $b := A$ *and* $d := C$*. Since* $\gamma \notin \Gamma_0(p)$ *we have* $C \not\equiv 0 \mod p$ *hence* $lC \equiv D \mod p$ *has a solution with* $0 \le l < p$*. We set* $c := lC - D$ *and* $a := lA - B$*. Hence* $c \equiv 0 \mod p$ *which means that* $\beta \in \Gamma_0(p)$.

b) Let $\mathcal{F} = SL_2(\mathbb{Z})\backslash\mathbb{H}$ denote the usual fundamental domain of $SL_2(\mathbb{Z})$ and set $\mathcal{F}_p := \bigcup_{l=0}^{p} \alpha_l\mathcal{F}$. Show that $\mathcal{F}_p$ is a fundamental domain of $\Gamma_0(p)$.

We have to show:

(i) If $\tau \in \mathbb{H}$*, there is a* $\beta \in \Gamma_0(p)$ *such that* $\beta\tau \in \overline{\mathcal{F}_p}$*.*
(ii) No two distinct points of $\mathcal{F}_p$ *are equivalent under* $\Gamma_0(p)$*.*

Proof of (i)***:***
Let $\tau \in \mathbb{H}$*. We already know that* $\mathcal{F}$ *is a fundamental domain for* $SL_2(\mathbb{Z})$ *hence we find* $\gamma \in SL_2(\mathbb{Z})$ *such that* $\gamma\tau \in \overline{\mathcal{F}}$*. By (the proof) of part a) we find* $\beta \in \Gamma_0(p)$ *such that* $\gamma^{-1} = \beta\alpha_l$ *for* $0 \le l \le p$*. Hence* $\beta^{-1}\tau = \alpha_l\gamma\tau \in \alpha_l\overline{\mathcal{F}} \subset \overline{\mathcal{F}_p}$*.*

Proof of (ii)***:***
Recall that if $\tau_1 = \gamma\tau_2$ *(for* $\tau_1, \tau_2 \in \mathcal{F}$*) and* $\gamma \in SL_2(\mathbb{Z})$*, then* $\tau_1 = \tau_2$ *and* $\gamma = \pm 1$*.*
Let $\tau_1, \tau_2 \in \mathcal{F}_p$ *with* $\beta\tau_1 = \tau_2$ *for* $\beta \in \Gamma_0(p)$*. We have to show that* $\tau_1 = \tau_2$*. Wlog there are three cases to consider:*

(a) $\tau_1, \tau_2 \in \mathcal{F}$
(b) $\tau_1 \in \mathcal{F}$*,* $\tau_2 \in \alpha_l \mathcal{F}$ *for* $0 \leq l < p$
(c) $\tau_1 \in \alpha_k \mathcal{F}$*,* $\tau_2 \in \alpha_l \mathcal{F}$ *for* $0 \leq k, l < p$

In case (a) *we have* $\tau_1 = \tau_2$ *since* $\beta \in SL_2(\mathbb{Z})$ *and we already know that* $\mathcal{F}$ *is a fundamental domain for* $SL_2(\mathbb{Z})$*.*
In case (b) *we have* $\tau_2 = \alpha_l \tau_2'$ *for* $\tau_2' \in \mathcal{F}$*. Hence we get* $\tau_1 = \beta^{-1}\alpha_l \tau_2'$ *and therefore* $\beta^{-1}\alpha_l = \pm 1$ *which implies* $\beta = \pm\alpha_l$*. But* $\beta = \pm\alpha_l \notin \Gamma_0(p)$ *for* $0 \leq l < p$ *as one easily checks. So we get a contradiction.*
In case (c) *we have* $\tau_1 = \alpha_k \tau_1'$*,* $\tau_2 = \alpha_l \tau_2'$ *for* $\tau_1', \tau_2' \in \mathcal{F}$*. Hence* $\beta\alpha_k\tau_1' = \alpha_l\tau_2'$ *and therefore (as in part* (b)*)* $\beta\alpha_k = \pm\alpha_l$*. So we get:*

$$\beta = \pm\alpha_l\alpha_k^{-1} = \pm ST^l T^{-k} S^{-1} = \pm ST^{l-k}S = \pm \begin{pmatrix} -1 & 0 \\ l-k & -1 \end{pmatrix}$$

Since $\beta \in \Gamma_0(p)$ *this requires* $l \equiv k \mod p$*. But since* $0 \leq l, k < p$ *this implies* $l = k$*. Therefore* $\beta = \pm 1$ *and* $\tau_1 = \tau_2$*. This completes the proof.*

c) Draw a picture of $\mathcal{F}_2$. What are the cusps of $\Gamma_0(2)$?

The fundamental domain $\mathcal{F}_2$ ***of*** $\Gamma_0(2)$***:***
By task $a)$ *and* $b)$ *we have:*

$$SL_2(\mathbb{Z}) = S\Gamma_0(2) \,\dot\cup\, ST\Gamma_0(2) \,\dot\cup\, \Gamma_0(2)$$
$$\mathcal{F}_2 = S\mathcal{F} \,\dot\cup\, ST\mathcal{F} \,\dot\cup\, \mathcal{F}$$

To get an idea how $\mathcal{F}_2$ *resp.* $S\mathcal{F}$ *and* $ST\mathcal{F}$ *look like first note that* $T\tau = \tau + 1$ *just shifts our fundamental domain by* 1 *and* $S\tau = \frac{-1}{\tau}$ *maps* $S\overline{\mathcal{F}}$ *and* $ST\overline{\mathcal{F}}$ *to a compact set inside or on the unit ball with non-negative imaginary part. It has three vertices and edges in both cases. To find the vertices we look at the images under the transformations* S *and* ST *of the vertices of* $\mathcal{F}$*. The vertices are:*

$$i\infty,\ \mu_3 = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \text{ and } \mu_3 + 1 = \frac{-1}{\overline{\mu}} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

We have:

$$S(i\infty) = 0,\ S\mu_3 = \mu_3 + 1,\ S(\mu_3+1) = \mu_3 \quad \text{and}$$
$$ST(i\infty) = 0,\ ST\mu_3 = S(\mu_3+1) = \mu_3,\ ST(\mu_3+1) = \frac{-1}{\mu_3+2} = -\frac{1}{2} + i\frac{\sqrt{3}}{6}$$

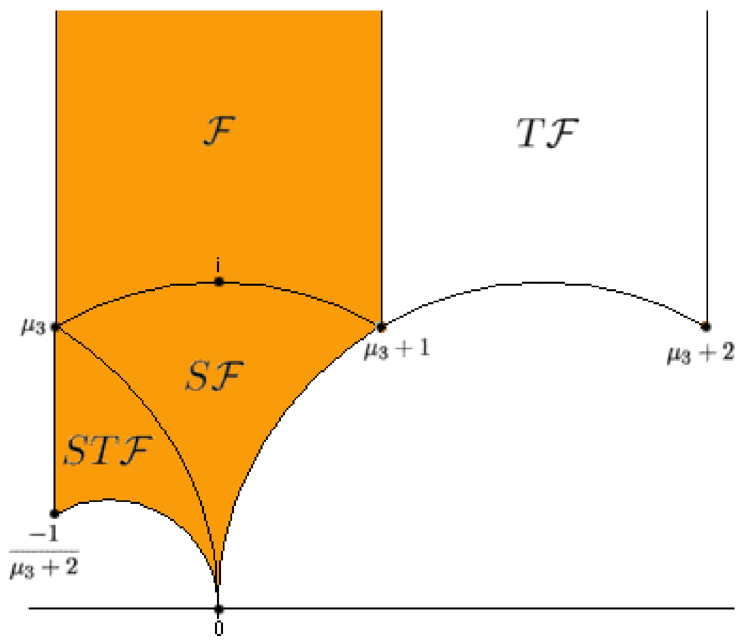
The last point is at one third of the height of μ_3*. This already gives us quite a good idea of how* $S\mathcal{F}$ *and* $ST\mathcal{F}$ *look like.*
To be more precise we can look at the images of the edges of $\mathcal{F}$ *and* $\mathcal{F}+1$ *under the transformation* $S\tau = \frac{-1}{\tau} = \frac{-x}{x^2+y^2} + i\frac{y}{x^2+y^2} = x' + iy'$*. Note that* $x'^2 + y'^2 = x^2 + y^2$*. The edges are (part of) the vertical lines* $x = \pm\frac{1}{2}, \frac{3}{2}$ *(denoted by* $l_{\pm\frac{1}{2}}$*,* $l_{\frac{3}{2}}$*) and the circles* $\{x^2 + y^2 = 1\}$ *resp.* $\{(x-1)^2 + y^2 = 1\}$ *centered at* 0 *with radius* 1 *(denoted by* $c_{0,1}$

resp. $c_{1,1}$*). Their images are again lines and circles since we have Moebius transformations. Clearly* e_1 *is mapped to itself. And for* e_2 *note that we have* $x^2 + y^2 = 2x$*. Hence for* $\tau' = x' + iy' \in Se_2$ *we have:* $x' = \frac{-x}{x^2+y^2} = \frac{-x}{2x} = \frac{-1}{2}$ *so* e_2 *is mapped to* $l_{-\frac{1}{2}}$*. Finally one can check that* $l_{\pm\frac{1}{2}}$ *are mapped to the circles* $c_{\mp 1,1}$ *and* $l_{\frac{3}{2}}$ *is mapped to the circle* $c_{\frac{1}{3},\frac{1}{3}}$*. Since we already know where the vertices are mapped it is easy to see what the exact images of the edges are. Below is a picture of* $\mathcal{F}_2$*. See:*

`http://www.math.lsu.edu/~verrill/fundomain/index2.html`

for more images of fundamental domains of congruence subgroups.

Picture of $\mathcal{F}_2$ ***:***



The cusps of $\Gamma_0(2)$***:***
We have to determine $\Gamma_0(2)\backslash\mathbb{Q} \cup (i\infty)$*. Let* $r \in \mathbb{Q}$*. There are two cases:*

(i) $r = \frac{a}{c}$ *with* $\gcd(a, c) = 1$ *and* c *even*

(ii) $r = \frac{b}{d}$ *with* $\gcd(b, d) = 1$ *and* d *odd*

In case (i) *we can extend any such* a, c *to a matrix* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$*.*

In case (ii) *we can also extend* b, d *to a matrix* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$*.*

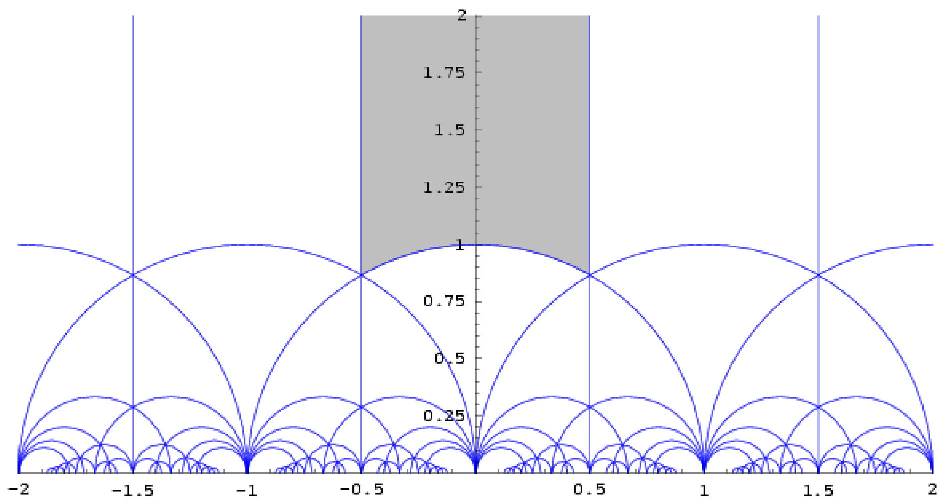
Furthermore in case (i) *we have* $\gamma(i\infty) = \frac{a}{c} = r$ *and in case* (ii) *we have* $\gamma 0 = \frac{b}{d} = r$*. Hence* $\mathbb{Q} \cup (i\infty) = \Gamma_0(2)(i\infty) \cup \Gamma_0(2)0$*. Since* a *is always odd we have* $\gamma(i\infty) \neq 0$ *for* *any* $\gamma \in \Gamma_0(2)$. *So* $i\infty$ *and* 0 *are* $\Gamma_0(2)$*-inequivalent, hence:*

$$\Gamma_0(2)\backslash\left(\mathbb{Q} \cup (i\infty)\right) = \Gamma_0(2)(i\infty) \,\dot{\cup}\, \Gamma_0(2)0$$

So $\Gamma_0(2)$ *has 2 cusps* $i\infty$ *and* 0. *This can also be seen from the picture of* $\mathcal{F}_2$.

Remark:

$\gamma\mathcal{F}$ is again a $SL_2(\mathbb{Z})$-equivalent fundamental domain for any $\gamma \in SL_2(\mathbb{Z})$ but it is not necessarely equivalent for subgroups of $SL_2(\mathbb{Z})$. $S\mathcal{F}$ and $ST\mathcal{F}$ for example are $SL_2(\mathbb{Z})$-equivalent but they are not $\Gamma_0(2)$-equivalent. If we divide $\mathbb{H}$ into $SL_2(\mathbb{Z})$-equivalent domains (up to borders) we get the following picture:



In general any subgroup G of $SL_2(\mathbb{Z})$ has a fundamental domain that is a union of such domains. They are given by $\gamma\mathcal{F}$ for γ representatives of $G\backslash SL_2(\mathbb{Z})$. Note that this is a finite group in the case of congruence subgroups and the number of needed domains is equal to the index.