Solution sheet 2

We first recall the Fourier expansions of the first few Eisenstein series:

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q(z)^n \qquad E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q(z)^n$$
$$E_6(z) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q(z)^n \qquad E_8(z) = 1 + 480\sum_{n=1}^{\infty}\sigma_7(n)q(z)^n$$
$$E_{10}(z) = 1 - 264\sum_{n=1}^{\infty}\sigma_9(n)q(z)^n \qquad E_{12}(z) = 1 + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q(z)^n$$
$$E_{14}(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_{13}(n)q(z)^n$$

where $\sigma_{k-1}(n) := \sum_{\substack{d|n \\ d\geq 1}} d^{k-1}$.

1. Let q denote the function from $\mathbb{H}$ to the punctured open unit ball defined by $q(z) := e^{2\pi i z}$.

The function $\Delta : \mathbb{H} \to \mathbb{C}$ that sends $z \in \mathbb{H}$ to $\Delta(z) := (2\pi)^{12}q(z)\prod_{n=1}^{\infty}(1-q(z)^n)^{24}$ is called the *discriminant function*.

Moreover, the function $\tau : \mathbb{N}_{>0} \to \mathbb{Z}$ defined trough the series

$$\sum_{n=1}^{\infty}\tau(n)q^n := q\prod_{n=1}^{\infty}(1-q^n)^{24} = q - 24q^2 + 252q^3 + O(q^4) \in \mathbb{Z}[[q]]$$

is called the *Ramanujan τ-function*.

a) Show that $\frac{1}{2\pi i}\frac{d}{dz}log\Delta(z) = E_2(z)$ and conclude that $\Delta \in S_{12}$.

Proof: As $|q(z)| < 1$ for any $z \in \mathbb{H}$, the product converges locally uniformly and defines a holomorphic and nowhere vanishing function on $\mathbb{H}$. We may therefore compute its logarithmic derivative. We have

$$\frac{1}{2\pi i}\frac{d}{dz} = \frac{1}{2\pi i}\frac{dq}{dz}\frac{d}{dq} = q\frac{d}{dq} \tag{1}$$

and therefore make the computaion

$$
\begin{aligned}
q\frac{d}{dq}\log\left(q\prod_{n=1}^{\infty}(1-q^n)^{24}\right) &= q\left(\frac{1}{q}+\sum_{n=1}^{\infty}\frac{-24nq^{n-1}(1-q^n)^{23}}{(1-q^n)^{24}})\right) \\
&= 1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} = 1-24\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}nq^nq^{nm} \\
&= 1-24\sum_{n,m=1}^{\infty}nq^{nm} = 1-24\sum_{n=1}\left(\sum_{d|n}d\right)q^n.
\end{aligned}
$$

As this sum is the Fourier expansion of E_2, we get

$$\frac{1}{2\pi i}\frac{d}{dz}\log\Delta(z) = E_2(z).$$

By its definition, Δ vanishes at infinity. In order to see that $\Delta \in S_{12}$, it thus only remains to be shown that $\Delta|[\gamma]_{12} = \Delta$ for any $\gamma \in SL_2(\mathbb{Z})$. Using basic differentiation rules and the properties of E_2 discussed on the first exercise sheet, we get for any such γ that

$$
\begin{aligned}
\frac{1}{2\pi i}\frac{d}{d\tau}\log\left(\Delta|[\gamma]_{12}(\tau)\right) &= -12\frac{1}{2\pi i}\frac{d}{d\tau}\log(j(\gamma,\tau)) + \frac{1}{2\pi i}\frac{d}{d\tau}\log(\Delta(\gamma\tau)) \\
&= -\frac{12c}{j(\gamma,\tau)2\pi i} + j(\gamma,\tau)^{-2}\left(\frac{1}{2\pi i}\frac{d}{d\tau}\log(\Delta(\cdot))\right)(\gamma\tau) \\
&= -\frac{12c}{j(\gamma,\tau)2\pi i} + j(\gamma,\tau)^{-2}E_2(\gamma\tau) \\
&= \left(E_2|[\gamma]_2\right)(\tau) - \frac{12c}{j(\gamma,\tau)2\pi i} \\
&= E_2(\tau) = \frac{1}{2\pi i}\frac{d}{d\tau}\log\left(\Delta(\tau)\right).
\end{aligned}
$$

Therefore $\Delta|[\gamma]_{12} = C(\gamma)\Delta$ for some $0 \neq C(\gamma) \in \mathbb{C}$. We claim that $C(\gamma) = 1$ for any $\gamma \in SL_2(\mathbb{Z})$. The fact that the slash operator defines a group action immediately implies that the association $\gamma \mapsto C(\gamma)$ is a group homomorphism from $SL_2(\mathbb{Z})$ to the multiplicative group of $\mathbb{C}$. It therefore suffices to show the claim for the cases in which γ is one of the generators T and S of $SL_2(\mathbb{Z})$. We have $C(T) = 1$ since q is invariant under $z \mapsto Tz = z+1$. Moreover, $\Delta|[S](z) = z^{-12}\Delta\left(\frac{-1}{z}\right)$ and thus $\Delta(i) = C(S)\Delta(i)$. Since $\Delta(i) \neq 0$, we get $C(S) = 1$.

b) Show that $\Delta = \frac{(2\pi)^{12}}{1728}\left(E_4^3 - E_6^2\right)$ and derive relations expressing τ in terms of σ_3 and σ_5.

Proof: The functions Δ and $E_4^3 - E_6^2$ are both non-zero and elements of S_{12}. As S_{12} is one dimensional, there exists some $c \in \mathbb{C}$ with $\Delta = c\left(E_4^3 - E_6^2\right)$. In order to determine c we compare the respective first Fourier coefficient of these functions. For Δ this is $(2\pi)^{12}\tau(1) = (2\pi)^{12}$ and for $E_4^3 - E_6^2$ this is $3\cdot 240 - 2\cdot(-504) = 1728$. Hence $c = \frac{(2\pi)^{12}}{1728}$.

Relation:

We have

$$1728 \sum_{n=1}^{\infty} \tau(n) q^n = \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n\right)^3 - \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n\right)^2 .$$

Comparing coefficients and dividing by 144 thus yields

$$\begin{aligned} 12\tau(n) = \quad & 5\sigma_3(n) \\ & + 1200 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) \\ & + 96000 \sum_{r=1}^{n-1} \sum_{m=1}^{r-1} \sigma_3(m)\sigma_3(r-m)\sigma_3(n-r) \\ & + 7\sigma_5(n) \\ & - 1764 \sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n-m) \end{aligned}$$

c) Show that $E_{12} - E_6^2 = c\Delta$ with $c = (2\pi)^{-12} 2^6 3^5 7^2 \frac{1}{691}$ and derive relations expressing τ in terms of σ_{11} and σ_5. Use this to prove Ramanujan's famous congruence relation

$$\tau(n) \equiv \sigma_{11}(n) \mod 691$$

for any $n \geq 1$.

Proof:

For the same reason as before there exists some $c \in \mathbb{C}$ such that $E_{12} - E_6^2 = c\Delta$. The first Fourier coefficient of $E_{12} - E_6^2$ is $\frac{65520}{691} - 2 \cdot (-504) = \frac{762048}{691} = 2^6 3^5 7^2 \frac{1}{691}$. Hence c is as stated above.

Relation:

$$2^6 3^5 7^2 \sum_{n=1}^{\infty} \tau(n) q^n = 691 \left(1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n\right) - 691 \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n\right)^2$$

Comparing coefficients yields

$$2^6 3^5 7^2 \tau(n) = 65520\sigma_{11}(n) + 691 \cdot 2 \cdot 504\sigma_5(n) - 691 \cdot 504^2 \sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n-m)$$

Dividing by 1008 and reducing modulo 691 gives

$$756\tau(n) \equiv 65\tau(n) \equiv 65\sigma_{11}(n) \mod 691$$

Hence $\tau(n) \equiv \sigma_{11}(n) \mod 691$.

2. **a)** Show that $E_4^2 = E_8$ and $E_4 E_6 = E_{10}$ and $E_6 E_8 = E_{14}$ and derive relations expressing

- σ_7 in terms of σ_3
- σ_9 in terms of σ_3 and σ_5
- σ_{13} in terms of σ_5 and σ_7.

Proof: By the structure theorem, the $\mathbb{C}$-vector spaces M_8, M_{10} and M_{14} are one dimensional. We thus have $E_4^2 = cE_8$, $E_4 E_6 = dE_{10}$ and $E_6 E_8 = eE_{14}$ for some $c, d, e \in \mathbb{C}$. Any of the E_k's and their powers have constant Fourier coefficient equal to 1. We therefore get $c = d = e = 1$ by comparing the constant Fourier coefficients on both sides.

Relations:

$$\left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n\right)^2 = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n$$
$$\left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n\right)\left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n\right) = 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n$$
$$\left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n\right)\left(1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n\right) = 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n$$

By comparison of coefficients we get:

$$\sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) = \sigma_7(n)$$
$$10\sigma_3(n) - 21\sigma_5(n) - 5040 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_5(n-m) = -11\sigma_9(n)$$
$$-21\sigma_5(n) + 20\sigma_7(n) - 10080 \sum_{m=1}^{n-1} \sigma_5(m)\sigma_7(n-m) = -\sigma_{13}(n)$$

b) Let $f \in M_k$ and set

$$g(z) := \frac{1}{2\pi i} \frac{d}{dz} f(z) - \frac{k}{12} E_2(z) f(z)$$

for any $z \in \mathbb{H}$. Show that this defines a modular form $g \in M_{k+2}$ and that $g \in S_{k+2}$ if and only if $f \in S_k$.

Proof: We have that g is holomorphic since $\frac{d}{d\tau} f$, E_2 and f are. Using (1) and the Fourier expansions of f and E_2 we moverover see that g is holomorphic at $i\infty$. It remains to show that $g|[\gamma]_{k+2} = g$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. First recall the following facts:

$$\frac{d}{d\tau}(\gamma\tau) = j(\gamma, \tau)^{-2} \qquad \frac{d}{d\tau} j(\gamma, \tau) = c$$
$$E_2 f|[\gamma]_{k+2} = E_2|[\gamma]_2 f|[\gamma]_k \qquad (f|[\gamma]_{k+2})(\tau) = j(\gamma, \tau)^{-2} f(\tau)$$

From these facts, the basic differentiation rules and the properties of E_2 known from the first exercise sheet we get

$$
\begin{aligned}
&(g|[\gamma]_{k+2})(\tau) = \frac{1}{2\pi i} j(\gamma,\tau)^{-(k+2)} f'(\gamma\tau) - \frac{k}{12}(E_2 f|[\gamma]_{k+2})(\tau) \\
&= \frac{1}{2\pi i} j(\gamma,\tau)^{-k} \frac{d}{d\tau} f(\gamma\tau) - \frac{k}{12}(E_2|[\gamma]_2)(\tau)(f|[\gamma]_k)(\tau) \\
&= \left(\frac{1}{2\pi i}\frac{d}{d\tau}(f|[\gamma]_k)(\tau) - \frac{1}{2\pi i}\left(\frac{d}{d\tau} j(\gamma,\tau)^{-k}\right) f(\gamma\tau)\right) - \frac{k}{12}(E_2|[\gamma]_2)(\tau) f(\tau) \\
&= \frac{1}{2\pi i}\frac{d}{d\tau} f(\tau) + \frac{kc}{j(\gamma,\tau)2\pi i}(f|[\gamma]_k)(\tau) - \frac{k}{12}(E_2|[\gamma]_2)(\tau) f(\tau) \\
&= \frac{1}{2\pi i}\frac{d}{d\tau} f(\tau) - \frac{k}{12}\left((E_2|[\gamma]_2)(\tau) - \frac{12c}{j(\gamma,\tau)2\pi i}\right) f(\tau) \\
&= \frac{1}{2\pi i}\frac{d}{d\tau} f(\tau) - \frac{k}{12} E_2(\tau) f(\tau) = g(\tau).
\end{aligned}
$$

Hence $g \in M_{k+2}$. Note that $f'(i\infty) = 0$ by (1) and $E_2(i\infty) = 1$. So $g(i\infty) = -\frac{k}{12} f(i\infty)$ therefore $f \in S_k$ if and only if $g \in S_{k+2}$.

c) Compute g explicitly for the cases where f is Δ or E_4 or $f = E_6$ and derive relations expressing

- σ_5 in terms of σ_1 and σ_3
- σ_7 in terms of σ_1 and σ_5.

Proof: If $f = \Delta \in S_{12}$, then $g \in S_{14} = \{0\}$ hence $g = 0$. Note that this is equivalent to the expression in task $1a$). So task $2b$) for $f = \Delta$ and $g = 0$ could be used to proof task $1a$).

If $f = E_4 \in M_4$, then $g \in M_6 = \mathbb{C}E_6$ hence $g = cE_6$ for some $c \in \mathbb{C}$. Since $E_6(i\infty) = 1$, we have $c = g(i\infty) = -\frac{1}{3}E_2(i\infty)E_4(i\infty) = -\frac{1}{3}$. Therfore $g = \frac{1}{3}E_6$.

If $f = E_6 \in M_6$, then $g \in M_8 = \mathbb{C}E_8$ and hence $g = cE_8$ for some $c \in \mathbb{C}$. Since $E_8(i\infty) = 1$, we have $c = g(i\infty) = -\frac{1}{2}E_2(i\infty)E_6(i\infty) = -\frac{1}{2}$ and hence $g = \frac{1}{2}E_8$.

Relations: Using the above results and (1) we get that

$$
\begin{aligned}
(1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n) =& -3q\frac{d}{dq}(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n) \\
&+ (1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n)(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n) \\
(1 + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n) =& -2q\frac{d}{dq}(1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n) \\
&+ (1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n)(1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n).
\end{aligned}
$$

Hence by comparison coefficients and dividing by 24 we get for any $n > 1$ that

$$21\sigma_5(n) = (30n - 10)\sigma_3(n) + \sigma_1(n) + 240 \sum_{m=1}^{n-1} \sigma_1(m)\sigma_3(n-m)$$
$$20\sigma_7(n) = (42n - 21)\sigma_5(n) + \sigma_1(n) + 504 \sum_{m=1}^{n-1} \sigma_1(m)\sigma_5(n-m).$$

3. We set $\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\} \subset \mathrm{SL}_2(\mathbb{Z})$. This is the stabilizer of infinity in any congruence subgroup of the form $\Gamma_0(N)$ and $\Gamma_1(N)$, where $N \geq 1$.

a) Find a system of representatives for $\Gamma_\infty \backslash \Gamma_0(N)$ for any $N \geq 1$.

Proof: For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have $ad - bc = 1$ and thus $\gcd(c, d) = 1$. We may therefore consider the map

$$\phi : \mathrm{SL}_2(\mathbb{Z}) \to \{(x, y) \in \mathbb{Z}^2 \,|(x, y) = (0, 1) \text{ or } \gcd(x, y) = 1, x \geq 0\}$$

that sends any $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to (c, d) if $c > 0$, to $(-c, -d)$ if $c < 0$ and to $(0, 1)$ if $c = 0$. By Bézout's Lemma, it is surjective. Moreover, it factors through $\Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})$ as is immediately checked. We claim that the factorization is bijective. Consider therefore any $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$ with $\phi(\gamma) = \phi(\gamma') = (x, y)$. Multiplying γ or γ' by (-1) if necessary, we may assume without loss of generality that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma' = \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$ for some $a, a', b, b', c, d \in \mathbb{Z}$. Using $a'd - b'c = ad - bc = 1$ we get

$$\gamma\gamma'^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b' \\ -c & a' \end{pmatrix} = \begin{pmatrix} 1 & a'b - ab' \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty$$

as needed. Moreover, it follows immediately from the definitions that the factorization restricts to a bijection

$$\Gamma_\infty \backslash \Gamma_0(N) \to \{(x, y) \in \mathbb{Z}^2 \,|(x, y) = (0, 1) \text{ or } \gcd(x, y) = 1, x \geq 0, x \in (N)\} =: I_0(N).$$

A system of representatives for $\Gamma_\infty \backslash \Gamma_0(N)$ can thus be found by the choice of an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ for any $(c, d) \in I_0(N)$.

b) Let $k \geq 4$ be an even integer and $N \geq 1$. For any $z \in \mathbb{H}$ we set

$$E_{k,N}(z) := \sum_{[\gamma] \in \Gamma_\infty \backslash \Gamma_0(N)} 1|[\gamma]_k(z) = \sum_{[\gamma] \in \Gamma_\infty \backslash \Gamma_0(N)} j(\gamma, z)^{-k} \quad \text{and}$$
$$G_{k,N}(z) := \frac{1}{2} \sideset{}{'}\sum_{m,n \in \mathbb{Z}} \frac{1}{(Nmz + n)^k}.$$

Show that this this defines modular forms in $M_k(\Gamma_0(N))$ with $G_{k,N} = \zeta(k)E_{k,N}$.

The modular form $E_{k,N}$ is called the *(normalised) Eisenstein series* of weight k and level N.

Proof of the first part: Using part a) we may rewrite

$$E_{k,N}(z) = \sum_{(c,d)\in I_0(N)} \frac{1}{(cz+d)^k}.$$

Since $k \geq 4$, this series converges absolutely and locally uniformly on $\mathbb{H}$. Hence $E_{k,N}$ is holomorphic on $\mathbb{H}$. We have

$$\begin{aligned}
G_{k,N}(z) &= \frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(Nmz+n)^k} = \frac{1}{2}\sum_{r=1}^{\infty} \sum_{\substack{m,n\in\mathbb{Z}\\ gcd(mN,n)=r}}{}' \frac{1}{r^k\left(N\frac{m}{r}z+\frac{n}{r}\right)^k} \\
&= \left(\sum_{r=1}^{\infty} r^k\right)\left(\frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\ gcd(mN,n)=1}}{}' \frac{1}{(Nmz+n)^k}\right) = \zeta(k) \sum_{(c,d)\in I_0(N)} \frac{1}{(cz+d)^k} \\
&= \zeta(k)E_{k,N}(z),
\end{aligned}$$

where in the second last step we have used that k is even.

It follows immediately from the definition that $E_{k,N}$, and thus also $G_{k,N}$, is invariant under the suitable slash operator. Moreover, $E_{k,N}$, and thus also $G_{k,N}$, is holomorphic at $i\infty$: For any $(0,1) \neq (c,d) \in I_0(N)$ the term $\frac{1}{(cz+d)^k}$ converges to 0 as z tends to $i\infty$ and thus only the constant term 1 induced by $(0,1) \in I_0(N)$ contributes to the value $E_{k,N}(i\infty)$ which is therefore 1.

Let us finally check that $E_{k,N}$, and thus also $G_{k,N}$, is in fact holomorphic at *any* cusp. In order to do so, it is enough to show for any $\sigma = \begin{pmatrix} v & w \\ x & y \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ that $E_{k,N}|[\sigma]_k$ is holomorphic at the cusp $i\infty$. We have

$$\begin{aligned}
E_{k,N}|[\sigma]_k &= \sum_{[\gamma]\in\Gamma_\infty\backslash\Gamma_0(N)} 1|[\gamma\sigma]_k(z) = \sum_{[\gamma]\in\Gamma_\infty\backslash\Gamma_0(N)} j(\gamma\sigma,z)^{-k} \\
&= \sum_{(c,d)\in I_0(N)} \frac{1}{((cv+dx)z+(cw+dy))^k}.
\end{aligned}$$

Letting z tend to $i\infty$, all of these summands tend to 0 except the ones for which $cv+dx = 0$. It is therefore enough to show that there are only finitely many $(c,d) \in I_0(N)$ that satisfy $cv+dx = 0$. However, this follows immediately from the unique factorization in $\mathbb{Z}$, the condition $\gcd(c,d) = 1$ and the fact that $(v,x) \neq (0,0)$.

c) Let $k \geq 0$. For any positive integers N and M such that N divides M we consider the *trace operator*

$$tr_N^M : M_k(\Gamma_0(M)) \to M_k(\Gamma_0(N))$$

defined by

$$tr_N^M(f) := \sum_{[\gamma]\in\Gamma_0(M)\backslash\Gamma_0(N)} f|[\gamma]_k$$

for any $f \in M_k(\Gamma_0(M))$.

Check that it is indeed well-defined and maps $S_k(\Gamma_0(M))$ to $S_k(\Gamma_0(N))$.

Show moreover that $tr_N^M(G_{k,M}) = G_{k,N}$ and in particular, that $tr_1^M(G_{k,M}) = G_k$.

Proof of the first part: Consider any $f \in M_k(\Gamma_0(M))$ and any $\gamma \in \Gamma_0(N)$. Then $f|[\gamma]_k = f|[\alpha\gamma]_k$ for any $\alpha \in \Gamma_0(M)$ since the automorphy factor of α is 1. Hence $f|[\gamma]_k$ does not depend on the representative γ of $[\gamma] \in \Gamma_0(M)\backslash\Gamma_0(N)$. Moreover $f|[\gamma]_k$ is holomorphic as f is. We moreover have for any $\gamma' \in \Gamma_0(N)$ that

$$\begin{aligned} tr_N^M(f)|[\gamma']_k &= \left(\sum_{[\gamma]\in\Gamma_0(M)\backslash\Gamma_0(N)} f|[\gamma]_k \right) \Bigg| [\gamma']_k = \sum_{[\gamma]\in\Gamma_0(M)\backslash\Gamma_0(N)} f|[\gamma\gamma']_k \\ &= \sum_{[\gamma'']\in\Gamma_0(M)\backslash\Gamma_0(N)} f|[\gamma'']_k = tr_N^M(f) \end{aligned}$$

Thus tr_N^M is well defined.

Remark

If $f \in S_k(\Gamma_0(M))$, then $f|[\alpha]_k$ vanishes at infinity for any $\alpha \in SL_2(\mathbb{Z})$. Hence each term in the sum

$$tr_N^M(f)|[\alpha]_k = \sum_{[\gamma]\in\Gamma_0(M)\backslash\Gamma_0(N)} f|[\gamma\alpha]_k$$

vanishes at infinity. Thus $tr_N^M(f) \in S_k(\Gamma_0(N))$.

Traces of Eisenstein series: By the same argument as above we get

$$\begin{aligned} tr_N^M(E_{k,M}) &= \sum_{[\gamma']\in\Gamma_0(M)\backslash\Gamma_0(N)} \left(\sum_{[\gamma]\in\Gamma_\infty\backslash\Gamma_0(M)} 1|[\gamma]_k \right) \Bigg| [\gamma']_k \\ &= \sum_{[\gamma']\in\Gamma_0(M)\backslash\Gamma_0(N)} \sum_{[\gamma]\in\Gamma_\infty\backslash\Gamma_0(M)} 1|[\gamma\gamma']_k \\ &= \sum_{[\gamma']\in\Gamma_0(M)\backslash\Gamma_0(N)} \sum_{[\gamma'']\in\Gamma_\infty\backslash(\Gamma_0(M)\gamma')} 1|[\gamma'']_k \\ &= \sum_{[\gamma'']\in\Gamma_\infty\backslash\left(\bigcup_{\gamma'\in\Gamma_0(M)\backslash\Gamma_0(N)} \Gamma_0(M)\gamma'\right)} 1|[\gamma'']_k \\ &= \sum_{[\gamma'']\in\Gamma_\infty\backslash\Gamma_0(N)} 1|[\gamma'']_k = E_{k,N} \end{aligned}$$

So of course we also have $tr_N^M(G_{k,M}) = G_{k,N}$ and $tr_1^N(G_{k,N}) = G_{k,1} = G_k$.

4. **a)** Prove that

$$\frac{\sin(z)}{z} = \prod_{n\geq 1}(1-\frac{z^2}{n^2\pi^2})$$

for any $z \in \mathbb{C}$.

Hint: Use that

$$\frac{\sin(z)}{z} = \frac{e^{iz}-e^{-iz}}{2iz} = \lim_{n\to\infty} p_n(z),$$

where

$$p_n(z) := \frac{(1+\frac{iz}{n})^n - (1-\frac{iz}{n})^n}{2iz}$$

and show that

$$p_n(z) = \prod_{1\leq k\leq p}\left(1-\frac{z^2}{n^2}\left(\frac{1+\cos(\frac{2k\pi}{n})}{1-\cos(\frac{2k\pi}{n})}\right)\right) \tag{2}$$

whenever $n = 2p+1$ for some integer p.

Proof: The following proof is based on Eberlein's article [1].

We first fix an odd positive integer n and establish the product formula for p_n. Note that an element $a \in \mathbb{C}$ is a zero of $(1+X)^n - (1-X)^n = 0$ if and only if there exists an n-th root of unity ω such that $(1+a) = \omega(1-a)$ or, equivalently, $a = \frac{\omega-1}{\omega+1}$. Note further that the zeroes of p_n are precisely the non-zero zeroes of $(1+\frac{iz}{n})^n - (1-\frac{iz}{n})^n$. These are thus the $\alpha_k := \frac{n}{i}\frac{e^{i\vartheta_k}-1}{e^{i\vartheta_k}+1}$, where $1 \leq k < n$ and $\vartheta_k := \frac{2\pi k}{n}$.

As n is odd, we may pair α_k with α_{n-k} for any $1 \leq k < n$. Using that $e^{i\vartheta_{n-k}} = e^{-i\vartheta_k}$ and that $cos(\vartheta_k) = \frac{e^{i\vartheta_k}+e^{-i\vartheta_k}}{2}$ we compute

$$(z-\alpha_k)(z-\alpha_{n-k}) = z^2 - n^2\frac{1+\cos(\vartheta_k)}{1-\cos(\vartheta_k)}.$$

Therefore the zeroes of p_n coincide with the zeroes of the product in (2). As these polynomial functions also have the same constant coefficient, they are equal.

In order to see that

$$\prod_{1\leq k\leq n}\left(1-\frac{z^2}{k^2\pi^2}\right) \overset{n\to\infty}{\longrightarrow} \frac{\sin(z)}{z}$$

for any $z \in \mathbb{C}$ it is enough to show that

$$\prod_{1\leq k\leq n}\left(1+\frac{z^2}{k^2\pi^2}\right) \overset{n\to\infty}{\longrightarrow} \frac{\sin(iz)}{iz}$$

for any $z \in \mathbb{C}$.

In order to prove the latter we set $g_n(z) := p_n(iz)$ and

$$g_{n,l}(z) := \prod_{1\leq k\leq l}\left(1+\frac{z^2}{n^2}\left(\frac{1+\cos(\frac{2k\pi}{n})}{1-\cos(\frac{2k\pi}{n})}\right)\right)$$

for any n of the form $n = 2p + 1$ and any $1 \leq l \leq p$.

Using the rule of Bernoulli and de l'Hopital twice in a row we find that $\frac{1+\cos(\vartheta_{k,n})}{n^2(1-\cos(\vartheta_{k,n}))}$ converges to $\frac{1}{k^2\pi^2}$ as n tends to infinity.

Since for any any positive real number x we have $g_{n,l}(x) \leq g_n(x)$, fixing l and letting n tend to infinity thus yields

$$\prod_{1 \leq k \leq l} \left(1 + \frac{x^2}{k^2\pi^2}\right) \leq \frac{\sin(ix)}{ix}.$$

On the other hand since $\frac{1}{\vartheta_{k,n}^2} \leq \frac{\cos(\vartheta_{k,n})}{1-\cos(\vartheta_{k,n})}$ for any $1 \leq k \leq l$, we have that

$$g_{2l+1}(x) \leq \prod_{1 \leq k \leq l} \left(1 + \frac{x^2}{k^2\pi^2}\right).$$

Letting l tend to infinity we thus obtain for any positive real number x that

$$\frac{\sin(ix)}{ix} = \prod_{k \geq 1} \left(1 - \frac{x^2}{k^2\pi^2}\right).$$

This proves what we want for a positive real number x. Finally the required identity for any complex number z follows from this case as follows.

If $n = 2p + 1$ and $1 \leq l \leq p$ and z is any complex number, then

$$\begin{aligned} |g_n(z) - g_{n,l}(z)| &= \left| g_{n,l}(z) \left(\prod_{l \leq k \leq p} \left(1 + \frac{z^2}{k^2\pi^2}\right) - 1 \right) \right| \\ &\leq g_{n,l}(|z|) \left(\prod_{l \leq k \leq p} \left(1 + \frac{|z|^2}{k^2\pi^2}\right) - 1 \right) \\ &= g_n(|z|) - g_{n,l}(|z|). \end{aligned}$$

Fixing l and letting n tend to infinity we get

$$\left| \frac{\sin(iz)}{iz} - \prod_{1 \leq k \leq l} \left(1 + \frac{z^2}{k^2\pi^2}\right) \right| \leq \frac{\sin(i|z|)}{i|z|} - \prod_{1 \leq k \leq l} \left(1 + \frac{|z|^2}{k^2\pi^2}\right).$$

The case of positive real elements thus implies

$$\prod_{1 \leq k \leq l} \left(1 + \frac{z^2}{k^2\pi^2}\right) \stackrel{l \to \infty}{\longrightarrow} \frac{\sin(iz)}{iz}$$

as desired.

b) Use logarithmic differentiation in order to deduce Euler's identity

$$\pi\cot(\pi z) = \frac{1}{z} + 2z\sum_{n\geq 1}\frac{1}{z^2-n^2}.$$

Proof: It is immediately checked that the product $\prod_{k\geq 1}(1-\frac{z^2}{k^2\pi^2})$ converges locally uniformly. Outside the set of zeroes of that product, we may therefore differentiate

$$\log\left(\prod_{k\geq 1}\left(1-\frac{z^2}{k^2\pi^2}\right)\right) = \sum_{k\geq 1}\log\left(1-\frac{z^2}{k^2\pi^2}\right)$$

summandwise by a theorem of Weierstrass. This immediately yields the desired identity.

c) Compare the power series expansions of $z\pi\cot(\pi z)$ and of $\frac{z}{e^z-1} = \sum_{k\geq 0}\frac{B_k z^k}{k!}$ in order to prove Euler's formula

$$\zeta(2k) = \frac{(-1)^{k-1}2^{2k-1}B_{2k}}{(2k)!}\pi^{2k}$$

for the value of the Riemann zeta function ζ at any positive even integer $2k$.

Conclude that $\frac{\zeta(2k)}{\pi^{2k}}$ is rational.

Proof: For any $|z| < 1$ we have

$$\pi\cot(\pi z) = \pi i\frac{e^{\pi iz}+e^{-\pi iz}}{e^{\pi iz}-e^{-\pi iz}} = \pi i\frac{e^{2\pi iz}+1}{e^{2\pi iz}-1} = \pi i + \frac{2\pi i}{e^{2\pi iz}-1}$$

and thus

$$\sum_{n\geq 1}\frac{2z}{z^2-n^2} = \pi\cot(\pi z) - \frac{1}{z} = \pi i + \frac{1}{z}\left(\frac{2\pi iz}{e^{2\pi iz}-1} - 1\right) = \pi i + \sum_{j\geq 1}\frac{B_j(2\pi i)^j z^{j-1}}{j!}.$$

The formula now follows from computing the value at 0 of the $(2k-1)$-th derivative of this function once via the left and once via the right hand side of the previous equation. By the locally uniform convergence of the sum on the left hand side we may compute its derivatives summandwise. Using $\frac{2z}{z^2-n^2} = \frac{1}{z-n} - \frac{1}{z+n}$ we get inductively that

$$\left(\frac{2z}{z^2-n^2}\right)^{(j)} = (-1)^j j!\left(\frac{1}{(z-n)^{j+1}} + \frac{1}{(z+n)^{j+1}}\right)$$

for any $j \geq 0$. Thus the value at 0 of the $(2k-1)$-th derivative of our function may be written as $(2k-1)!(-2)\zeta(2k)$. Computing this value via the right hand side yields

$$(2k-1)!(-2)\zeta(2k) = \frac{(-1)^k 2^{2k}\pi^{2k}B_{2k}}{2k}\pi^{2k}.$$

Literatur

[1] Eberlein, W.F. *On Euler's product for the sine*, Journal of Mathematical Analysis and Applications 58 (1) (1977), 147-151.