Solutions 3

1. **a)** Show that $j : \Gamma\backslash\overline{\mathbb{H}} \to \mathbb{C} \cup \{\infty\}$ gives a bijection between $\Gamma\backslash\overline{\mathbb{H}}$ and the Riemann sphere $\mathbb{C} \cup \{\infty\}$. More precisely, j maps ∞ to ∞ and induces a bijection between $\Gamma\backslash\mathbb{H}$ and the complex plane $\mathbb{C}$.

Solution : For any $\alpha \in \mathbb{C}$, the difference function $d := (12)^3 g_2^3 - \alpha\Delta$ defines a modular form of weight 12. Hence, the valence formula reads

$$\frac{1}{2}\mathrm{ord}_i(d) + \frac{1}{3}\mathrm{ord}_\varrho(d) + \sum \mathrm{ord}_p(d) \;=\; 1,$$

with each order being a non-negative integer. It follows that there is exactly one point $p \in \Gamma\backslash\mathbb{H}$ such that $j(p) = \alpha$.

b) Let $\mathcal{F}$ be the standard fundamental domain for the action of Γ on $\overline{\mathbb{H}}$.

Find $j(i)$, $j(\varrho)$, and determine all $\tau \in \mathcal{F}$ such that $j(\tau) \in \mathbb{R}$. Then show that j maps the left half of $\mathcal{F}$ onto $\mathbb{H}$ and the right half of $\mathcal{F}$ onto the lower half plane.

Solution : Plugging the relations $i^2 = -1$ and $\varrho^3 = 1$ into the definition of Eisenstein series yields

$$G_6(i) \;=\; \sum_{m,n} \frac{1}{(mi + n(-i^2))^6} \;=\; \frac{1}{i^6}\sum_{m,n} \frac{1}{(m - ni)^6} \;=\; -G_6(i)$$

and similar relations for $G_4(\varrho)$ so that we conclude that $G_6(i) = 0$ and $G_4(\varrho) = 0$. Therefore, $j(i) = (12)^3$ and $j(\varrho) = 0$.

Next, we determine all $\tau \in \mathcal{F}$ such that $j(\tau)$ is real. The q-expansion of the j-function has the form

$$j \;=\; \frac{1}{q} + \sum_{n \geq 0} a_n q^n.$$

One can immediately read from this expansion the relation

$$\overline{j(\tau)} \;=\; j(-\overline{\tau}), \tag{1}$$

that is, points that are reflections of one another with respect to the imaginary axis will be mapped to conjugate values. In particular, all points in $\mathcal{F}$ that lie on the imaginary axis will be mapped to real values by the j-function. Secondly, all points in $\mathcal{F}$ that lie on the unit circle centered at the origin – that is, explicilty, all points on the circular arc connecting ϱ and i – satisfy $\tau\overline{\tau} = 1$ and hence

$$j(\tau) \;=\; \overline{j(-\overline{\tau})} \;=\; \overline{j(-1/\tau)} \;=\; \overline{j\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\tau\right)} \;=\; \overline{j(\tau)},$$

where the last equality follows from the modularity of j. Hence, all points on the circular arc between ϱ and i also map to real values. Thirdly, any point lying on the left vertical boundary of $\mathcal{F}$ is expressed in rectangular coordinates as $-1/2+iy$ for some $y \geq \sqrt{3}/2$, and it is immediate from plugging in these values in the q-expansion of j that

$$j(-1/2+iy) \;=\; -e^{2\pi y} + \sum_{n\geq 0} a_n(-1)^n e^{-2\pi n y} \in \mathbb{R}.$$

Let $\mathcal{C}$ denote the contour encircling the left half of $\mathcal{F}$, that is the closed path composed of the vertical half-line going from $i\infty$ to ϱ, the circular arc from ϱ to i and the vertical half-line from i back to $i\infty$. If one runs along $\mathcal{C}$, then the enclosed left half $\mathcal{F}_L$ of $\mathcal{F}$ is always to the left, so that under j, that region will be mapped to the left of the real axis, that is to the upper half plane. By the symmetry relation above, the right-half $\mathcal{F}_R$ of $\mathcal{F}$ is mapped to the lower half plane. Hence, by part (a), $j(\mathcal{C}) = \mathbb{R}$, $j(\mathcal{F}_L) = \mathbb{H}$, and $j(\mathcal{F}_R)$ is the lower half plane.

2. Given a lattice $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$, let

$$g_2 := g_2(L) = g_2(\omega_1,\omega_2) \;=\; 60\sum_{m,n}(m\omega_1+n\omega_2)^{-4},$$
$$g_3 := g_3(L) = g_3(\omega_1,\omega_2) \;=\; 140\sum_{m,n}(m\omega_1+n\omega_2)^{-6}.$$

These two functions g_2 and g_3 are called the invariants of L. Observe that $g_2^3 - 27g_3^2 \neq 0$.

Prove that given two complex numbers a_2 and a_3 satisfying $a_2^3 - 27a_3^2 \neq 0$, there exist complex numbers ω_1 and ω_2 such that ω_1/ω_2 is not real, and $g_2(\omega_1,\omega_2) = a_2$, $g_3(\omega_1,\omega_2) = a_3$.

Solution : Set $\alpha := (12)^3\frac{a_2^3}{a_2^3-27a_3^2}$. First note that given a point $z \in \mathbb{H}$, we can associate to it the lattice $L_z := \mathbb{Z}z \oplus \mathbb{Z}$. Then

$$g_2(L_z) \;=\; 60\sum_{m,n}\frac{1}{(mz+n)^4} \;=\; g_2(z),$$

and the same goes for $g_3(L_z) = g_3(z)$.

We know from Ex. 1 that there is exactly one point $z \in \mathcal{F}$ such that $j(z) = \alpha$. Moreover, we also know that if $a_2 = 0$, this point must be $z = \varrho$, and that if $a_3 = 0$, this point must be $z = i$. (The condition $a_2^3 - 27a_3^2 \neq 0$ insures that the two complex numbers can not be both zero.)

Assume first that $a_2 = 0$ and take $\lambda \in \mathbb{C}^\times$ such that $\lambda^6 = \frac{g_3(\varrho)}{a_3}$. (Note that $g_3(\varrho)$ is non-zero since we know from Ex. 1 that $g_2(\varrho) = 0$ and Δ is never zero.) Then

$$g_3(\lambda L_\varrho) \;=\; \frac{g_3(L_\varrho)}{\lambda^6} \;=\; a_3$$

and $g_2(\lambda L_\varrho) = a_2 = 0$. If $a_3 = 0$, the same argument shows that $g_2(L) = a_2$ and $g_3(L) = a_3$ for $L = \frac{g_2(i)}{a_2}L_i$.

Finally, suppose that $a_2 a_3 \neq 0$. Then $j(z) = \alpha$ is equivalent to

$$(12)^3 \frac{g_2^3(z)}{g_2^3(z) - 27 g_3^2(z)} \;=\; (12)^3 \frac{a_2^3}{a_2^3 - 27 a_3^2}$$

which is equivalent to

$$\frac{a_3^2}{a_2^3} \;=\; \frac{g_3^2(z)}{g_2^3(z)} \;=\; \frac{g_3^2(L_z)}{g_2^3(L_z)} \;=\; \frac{g_3^2(\lambda L_z)}{g_2^3(\lambda L_z)},$$

for any $\lambda \neq 0$. now, take $\lambda \in \mathbb{C}^\times$ such that $\lambda^2 = \frac{a_2}{a_3}\frac{g_3}{g_2}$. Then

$$\frac{a_3^2}{a_2^3} \;=\; \frac{g_3^2(\lambda L_z)}{g_2^3(\lambda L_z)} \;=\; \left(\frac{a_3}{a_2}\right)^2 \frac{1}{g_2(\lambda L_z)},$$

hence $g_2(\lambda L_z) = a_2$ and, similarly, one can show that $g_3(\lambda L_z) = a_3$. In terms of basis elements (ω_1, ω_2), we can then choose the pair $(\lambda z, \lambda)$.

3. Let $V = \{(z, w) \in \mathbb{C}^2 : z^3 - 27w^2 = 0\}$. Let S^3 denote the 3-sphere. Then $T = V \cap S^3$ is the trefoil knot.

Prove that the space of lattices $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ can be identified with the complement of the trefoil knot $S^3 \backslash T$.

Note : In fact, they are even diffeomorphic.

Solution : Each lattice can be written as $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ for a choice of basis (ω_1, ω_2). A unimodular lattice is a lattice Λ of covolume 1, that is, $\mathrm{vol}(\mathbb{R}^2/\Lambda) = 1$ or, equivalently, $\det(\omega_1|\omega_2) = 1$. (The notation $(\omega_1|\omega_2)$ refers to the two-by-two matrix with ω_1 and ω_2 as column vectors.)

We first show that the quotient $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ parametrizes the set of unimodular lattices. Any unimodular lattice can be seen to arise from an element of $\mathrm{SL}(2,\mathbb{R})$ via the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\mapsto\; \mathbb{Z}\begin{pmatrix} a \\ b \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} c \\ d \end{pmatrix} \;=: \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2.$$

This map then factors through the quotient $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ as can be checked by direct computation. This reflects the fact that if one takes another basis (ω_1', ω_2') for the unimodular lattice $\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ above, then the new basis elements can be expressed in terms of the old ones, i.e.

$$\begin{aligned} \omega_1' &= a\omega_1 + b\omega_2 \\ \omega_2' &= c\omega_1 + d\omega_2, \end{aligned}$$

with a, b, c, d integer coefficients and a simple computation would establish that $\det(\omega_1'|\omega_2') = (ad - bc)\det(\omega_1|\omega_2)$.

In a similar fashion, one can show that the quotient $\mathrm{PGL}(2,\mathbb{Z})\backslash\mathrm{PGL}(2,\mathbb{R})$ parametrizes all lattices up to homothety, that is the set of all equivalence classes $[\Lambda]$ where Λ is a lattice and $[\lambda\Lambda] = [\Lambda]$ for all $\lambda \in \mathbb{R}_{>0}$.

The claim is more transparent if we identify $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ with $\mathrm{PGL}(2,\mathbb{Z})\backslash\mathrm{PGL}(2,\mathbb{R})$. This identification is given by the map $\Lambda \mapsto [\Lambda]$. In fact, each equivalence class has a unimodular representative

$$\frac{1}{\sqrt{\mathrm{vol}(\mathbb{R}^2/\Lambda)}}\Lambda,$$

and for two images of this map such that $[\Lambda] = [\Lambda']$, there is a scalar $\lambda > 0$ such that $\Lambda' = \lambda\Lambda$. But because $\mathrm{vol}(\mathbb{R}^2/\Lambda') = \lambda^2\mathrm{vol}(\mathbb{R}^2/\Lambda)$ and both Λ, Λ' are taken to be unimodular, $\lambda = 1$.

By Exercise 2, we know that the map

$$\mathcal{L} \to \mathbb{C}^2 \setminus V, \quad \Lambda \mapsto (g_2(\Lambda), g_3(\Lambda))$$

defined on the set $\mathcal{L}$ of all lattices is surjective. Consider the composition with the projection to the 3-sphere, i.e.

$$\mathcal{L} \to S^3 \setminus K.$$

Then for two lattices Λ, $\Lambda' \in \mathcal{L}$ with the same image in $S^3 \setminus K$, there must be some positive scalar λ such that $\Lambda' = \lambda\Lambda$. We can conclude that $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R}) \cong S^3 \setminus K$.

4. Prove Picard's Theorem :

Every non-constant entire function attains every complex value with at most one exception.

Proof : We will prove the equivalent statement : Each entire function f that omits two distinct points $a, b \in \mathbb{C}$ is constant.

The idea of proof is as follows : Given an entire function g that is never 0 or $(12)^3$, the map $\exp(i(j^{-1} \circ g))$ is entire and bounded to the unit disk, hence constant by Liouville.

The function

$$g = (12)^3 \frac{f-a}{b-a}$$

is such a function ; it is entire and will never be either 0 or $(12)^3$. We can see the j-function $j : \mathbb{H} \to \mathbb{C}$ as an infinitely-sheeted branched covering map (indeed each $\alpha \in \mathbb{C}$ has preimage an infinite Γ-orbit), with branch points at $j^{-1}(0)$ and $j^{-1}((12)^3)$. Hence, the restriction to

$$j : \mathbb{H} \setminus \{j^{-1}(0), j^{-1}((12)^3)\} \to \mathbb{C} \setminus \{0, 1\}$$

defines an infinitely sheeted unbranched covering. Fix a branch for the multi-valued inverse function j^{-1}. Then the composition map $h := j^{-1} \circ g : \mathbb{C} \to \mathbb{H} \setminus \{\varrho, i\}$ can be analytically continued to all of $\mathbb{C}$, and this, by the Monodromy Theorem, as a single-valued analytic function, which we also denote h.

Now, the map $\varphi(z) = e^{ih(z)}$ is also entire but as $|\varphi(z)| = e^{-\mathrm{Im}(h(z))}$ and $h(z) \in \mathbb{H}$, it is bounded by the unit disk, and hence constant. It follows that h and hence g are constant, therefore $f = a + (b-a)g$ is a constant function.