Solutions 5

1. Let $a := (a(n))_{n\geq 1}$ be a sequence of complex numbers. We say that the sequence a is multiplicative if $a(mn) = a(m)a(n)$ for all coprime integers m, n (i.e. $\gcd(m,n) = 1$ for all $m, n \geq 1$). The sequence a is called completely multiplicative if $a(mn) = a(m)a(n)$ holds in general.

Let $\sigma_a \in \mathbb{R}$ be such that

$$L(s) := \sum_{n\geq 1} \frac{a(n)}{n^s}$$

converges absolutely on the half plane of convergence $H(a) := \{s \in \mathbb{C} |\ \mathrm{Re}(s) > \sigma_a\}$.

a) Show that if a is multiplicative, then

$$L(s) = \prod_p \left(\sum_{k\geq 0} \frac{a(p^k)}{p^{ks}} \right)$$

for all $s \in H(a)$.

Solution : Recall the fundamental theorem of arithmetic and consider the set $P(x, y)$ of all natural numbers whose prime decomposition $n = p_1^{k_1} \cdots p_m^{k_m}$ is such that each prime factor p_i is bounded by x and all powers k_i are bounded by y. Then

$$\begin{aligned}\sum_{n\in P(x,y)} \frac{a(n)}{n^s} &= \sum_{\substack{n=p_1^{k_1}\cdots p_m^{k_m} \\ p_i\leq x,\ k_i\leq y}} \frac{a(p_1^{k_1}\cdots p_m^{k_m})}{(p_1^{k_1}\cdots p_m^{k_m})^s} \\ &= \sum \left(\frac{a(p_1^{k_1})}{p_1^{k_1 s}} \cdots \frac{a(p_m^{k_m})}{p_m^{k_m s}} \right) = \prod_{p\leq x} \left(\sum_{k=0}^{y} \frac{a(p^k)}{p^{ks}} \right)\end{aligned}$$

where the second equality is obtained from the multiplicativity of a, and the third equality from reordering the summands. For $s \in H(a)$, the RHS is absoluetly convergent in x and y and the claim follows.

b) Show that if a is completely multiplicative, then

$$L(s) = \prod_p \frac{1}{1 - a(p)p^{-s}}$$

for all $s \in H(a)$.

Solution : As a is completely multiplicative, $a(p^k) = a(p)^k$ and the claim follows from the limit formula for the geometric series.

2. Let $f\ \mathbb{R}_+^\times \to \mathbb{C}$ be a continuous function such that $f(y)y^{s-1} \in L^1(\mathbb{R}_+^\times)$ for each

$$s \in \langle \alpha, \beta \rangle := \{s \in \mathbb{C} \mid \alpha < \mathrm{Re}(s) < \beta\}$$

the fundamental strip determined by $\alpha < \beta \in \mathbb{R} \cup \infty$. Its Mellin transform is defined by

$$\mathcal{M}(f)(s) := \int_0^\infty f(y) y^s \frac{dy}{y}$$

for all $s \in \langle \alpha, \beta \rangle$.

a) Show that $\mathcal{M}(f)$ is well-defined and holomorphic.

Solution : Set $g(y,s) := f(y)y^{s-1}$ and for $n \in \mathbb{Z}_{>1}$ $G_n(s) := \int_{\frac{1}{n}}^n g(y,s)dy$. Clearly $g(y,s)$ is holomorphic in s. Recall that for holomorphic functions $g(y,s)$ (in s) we have:

$$\frac{d}{d\bar{s}} \int_A g(y,s)dy = \int_A \frac{\partial}{\partial \bar{s}} g(y,s)dy = \int_A 0dy = 0$$

So clearly $G_n(s)$ is still holomorphic and $\lim_{n\to\infty} G_n(s) = \mathcal{M}_f(s) =: G(s)$.
Moreover the G_n's converges locally uniformly:
For $s \in K$ (K compact) we have $\alpha < c_1 < \mathrm{Re}(s) < c_2 < \beta$ for some constants c_1 and c_2. So we have the following estimate:

$$\begin{aligned} |G_n(s) - G(s)| &\le \int_0^{\frac{1}{n}} |g(y,s)|dy + \int_n^\infty |g(y,s)|dy \\ &= \int_0^{\frac{1}{n}} |f(y)|y^{\mathrm{Re}(s)-1}dy + \int_n^\infty |f(y)|y^{\mathrm{Re}(s)-1}dy \\ &\le \int_0^{\frac{1}{n}} |f(y)|y^{c_1-1}dy + \int_n^\infty |f(y)|y^{c_2-1}dy \end{aligned}$$

Where the last expression tends to 0 (for $n \to \infty$) by our assumptions ($c_1, c_2 \in <\alpha, \beta>$ and $g(y,s) \in L^1(\mathbb{R}^+)$ with respect to y). So we have a uniform bound for $|G_n(s) - G(s)|$. Hence the convergence is locally uniform. So the theorem of Weierstrass tells us that $G(s)$ is holomorphic.

b) Prove the following identities for $\mathcal{M}(f)$:

$$\begin{aligned} \mathcal{M}(y^\nu f(y))(s) &= \mathcal{M}(f(y))(s+\nu) \\ \mathcal{M}(f(\nu y))(s) &= \nu^{-s}\mathcal{M}(f(y))(s) \\ \mathcal{M}(f(y^\nu))(s) &= \frac{1}{\nu}\mathcal{M}(f(y))\left(\frac{s}{\nu}\right) \\ \mathcal{M}\left(\frac{1}{y} f\left(\frac{1}{y}\right)\right)(s) &= \mathcal{M}(f(y))(1-s) \\ \frac{d}{ds}\mathcal{M}(f(y))(s) &= \mathcal{M}(f(y)\log y)(s) \\ \mathcal{M}\left(\frac{d}{dy}f(y)\right)(s) &= -(s-1)\mathcal{M}(f(y))(s-1) \end{aligned}$$

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where $\nu > 0$.

Solution :

$$\mathcal{M}(y^\nu f(y))(s) = \int_0^\infty f(y) y^{s+\nu} \frac{dy}{y} = \mathcal{M}(f(y))(s+\nu)$$

$$\mathcal{M}(f(\nu y))(s) = \int_0^\infty f(\nu y) y^s \frac{dy}{y} \overset{(y'=y\nu)}{=} \nu^{-s} \int_0^\infty f(y') y'^s \frac{dy'}{y'} = \nu^{-s}\mathcal{M}(f(y))(s)$$

$$\mathcal{M}(f(y^\nu))(s) = \int_0^\infty f(y^\nu) y^s \frac{dy}{y} \overset{(y'=y^\nu)}{=} \frac{1}{\nu} \int_0^\infty f(y') y'^{\frac{s}{\nu}} \frac{dy'}{y'} = \frac{1}{\nu}\mathcal{M}(f(y))\left(\frac{s}{\nu}\right)$$

$$\mathcal{M}\left(\frac{1}{y} f\left(\frac{1}{y}\right)\right)(s) = \int_0^\infty \frac{1}{y} f\left(\frac{1}{y}\right) y^s \frac{dy}{y} \overset{\left(y'=\frac{1}{y}\right)}{=} \int_0^\infty y' f(y') y'^{-s} \frac{dy}{y} = \mathcal{M}(f(y))(1-s)$$

$$\frac{d}{ds}\mathcal{M}(f(y))(s) = \int_0^\infty f(y) \frac{d}{ds} y^s \frac{dy}{y} = \int_0^\infty f(y) \log y y^s \frac{dy}{y} = \mathcal{M}(f(y)\log y)(s)$$

$$\mathcal{M}\left(\frac{d}{dy} f(y)\right)(s) = \int_0^\infty \frac{d}{dy} f(y) y^s \frac{dy}{y} = f(y) y^{s-1}|_0^\infty - \int_0^\infty f(y)(s-1) y^{s-1} \frac{dy}{y}$$
$$= 0 - (s-1)\mathcal{M}(f(y))(s-1)$$

3. Recall the Gamma function $\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}$ defined for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. Prove that

a) The function $\Gamma(s)$ can be analytically continued to the whole complex plane into a meromorphic function whose poles are exactly non-positive integers and satisfies the functional equation $\Gamma(s+1) = \Gamma(s)$.

Solution : One can prove directly $\Gamma(s+1) = s\Gamma(s)$ using the definition of $\Gamma(s)$ and integration by parts ;

$$\Gamma(s+1) \;=\; \int_0^\infty e^{-t} t^s dt \;=\; \int_0^\infty e^{-t}(s t^{s-1}) dt \;=\; s\Gamma(s).$$

This allows to extend $\Gamma(s) = \frac{1}{s}\Gamma(s+1)$ to the whole complex s-plane, with simple poles at each non-positive integer $s = -n$, $n \in \mathbb{N}_0$.

b) Show that the meromorphic continuation satisfies $\Gamma(s) = \sum_{k=0}^\infty \frac{(-1)^k}{k!(k+s)} + \int_1^\infty e^{-y} y^s \frac{dy}{y}$ and conclude that $\mathrm{Res}(\Gamma, -n) = \frac{(-1)^n}{n!}$.

Solution :

$$\int_0^1 e^{-y} y^s \frac{dy}{y} = \int_0^1 \sum_{k=0}^\infty \frac{(-1)^k}{k!} y^k y^s \frac{dy}{y} = \sum_{k=0}^\infty \frac{(-1)^k}{k!} \int_0^1 y^{k+s} \frac{dy}{y}$$
$$= \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{y^{k+s}}{k+s}\bigg|_0^1 = \sum_{k=0}^\infty \frac{(-1)^k}{k!(k+s)}$$

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Hence $\Gamma(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+s)} + \int_1^{\infty} e^{-y} y^s \frac{dy}{y}$. Since the second part (the integral) is an entire function on $\mathbb{C}$ we only have to consider the first part (the sum) for the residues:

$$\operatorname{Res}(\Gamma, -n) = \lim_{s \to -n} (n+s) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+s)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \lim_{s \to -n} \frac{n+s}{k+s} = \frac{(-1)^n}{n!}$$

c) Prove the reflection formula $\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}$ and conclude that $\frac{1}{\Gamma(s)}$ is an entire function of s.

Solution : Set $g(s) := \Gamma(1-s)\Gamma(s) - \frac{\pi}{\sin(\pi s)}$. By part $a)$ we have that $g(s)$ is holomorphic on $\mathbb{C}\backslash\mathbb{Z}$. Let $-n \in \mathbb{Z}_{\leq 0}$. By task $3b$ we have:

$$\operatorname{Res}(\Gamma(1-s)\Gamma(s), -n) = \Gamma(n+1)\frac{(-1)^n}{n!} = (-1)^n = \operatorname{Res}\left(\frac{\pi}{\sin(\pi s)}, -n\right)$$

Let $n \in \mathbb{Z}_{>0}$. By task $3b$ we have:

$$\operatorname{Res}(\Gamma(1-s)\Gamma(s), n) = -\frac{(-1)^{n-1}}{(n-1)!}\Gamma(n) = (-1)^n = \operatorname{Res}\left(\frac{\pi}{\sin(\pi s)}, n\right)$$

So $g(s)$ has removable singularities at $s \in \mathbb{Z}$ and is therefore an entire function. Note that $\frac{1}{|sin(\pi s)|}$ is bounded for $|\Im(s)| > 1$. Since $s \in \mathbb{Z}$ are the only poles in $\mathbb{C}$ of the continious function $sin(\pi s)$ we also have that $\frac{1}{|sin(\pi s)|}$ is bounded in any compactum which does not contain elements of $\mathbb{Z}$. So $sin(\pi s)$ is bounded on $\mathbb{C}\backslash D$ for any region D containing $\mathbb{Z}$. So by equation

$$|\Gamma(s)| \leq \int_0^1 e^{-y} y^{\frac{1}{N}} \frac{dy}{y} + \int_1^{\infty} e^{-y} y^N \frac{dy}{y} \tag{1}$$

we have that $g(s)$ is bounded on $< \varepsilon, 1-\varepsilon >$ for any $\varepsilon > 0$. But by the functional equation $\Gamma(1-s) = \frac{1}{1-s}\Gamma(2-s)$ and the remarks above we see that $g(s)$ remains bounded on $\mathbb{C}\backslash D$. But the singularities at $s \in \mathbb{Z}$ are removable so $g(s)$ is bounded around $s \in \mathbb{Z}$ and thus bounded on $\mathbb{C}$ and therefore constant. Since $\lim_{y \to \infty} g(s) = 0$ we get $g(s) \equiv 0$.

By Euler's reflection formula and the fact that $\sin(\pi s)$ is entire we immediately get that $\Gamma(s)\Gamma(1-s) \neq 0$ on $\mathbb{C}$. So $\Gamma(s) \neq 0$ on $\mathbb{C}$. Hence $\frac{1}{\Gamma(s)}$ is entire.

d) Compute $\Gamma\left(\frac{1}{2}\right)$ and prove the duplication formula $\Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = 2^{\frac{1}{2}-2s}\sqrt{2\pi}\,\Gamma(2s)$.

Solution : By applying the reflection formula with $s = 1/2$, we obtain $\Gamma(1/2) = \sqrt{\pi}$. We define the **Beta function**:

$$B(r,s) := \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} = \int_0^1 z^{r-1}(1-z)^{s-1} dz$$

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The last equality follows from:

$$
\begin{aligned}
\Gamma(r)\Gamma(s) &= \int_0^\infty \int_0^\infty e^{-(u+v)} u^{r-1} v^{s-1} du dv \\
&\overset{(\sigma = u+v)}{=} \int_{\sigma \geq u \geq 0} e^{-\sigma} u^{r-1} (\sigma - u)^{s-1} du d\sigma \\
&\overset{(z = \frac{u}{\sigma})}{=} \int_{\substack{0 \leq z \leq 1 \\ u \geq 0}} e^{-\sigma} \left(z^{r-1}\sigma^{r-1}\right) \left((1-z)^{s-1}\sigma^{s-1}\right) (\sigma dz d\sigma) \\
&= \int_0^1 z^{r-1}(1-z)^{s-1} dz \int_0^\infty e^{-\sigma} \sigma^{r+s-1} d\sigma \\
&= \left(\int_0^1 z^{r-1}(1-z)^{s-1} dz \right) \Gamma(r+s)
\end{aligned}
$$

If we make the substitution $z = x^2$ we get:

$$
B(r,s) = \int_0^1 x^{2(r-1)}(1-x^2)^{s-1}(2x dx) = 2\int_0^1 x^{2r-1}(1-x^2)^{s-1} dx
$$

On the other hand if we set $r = s$ and make the substitution $z = \frac{1+x}{2}$ we get:

$$
\begin{aligned}
\frac{\Gamma(s)^2}{\Gamma(2s)} &= B(s,s) = \int_{-1}^1 \left(\frac{1+x}{2}\right)^{s-1} \left(1 - \frac{1+x}{2}\right)^{s-1} \left(\frac{1}{2} dx\right) \\
&= \frac{1}{2} \int_{-1}^1 \left(\frac{1+x}{2}\right)^{s-1} \left(\frac{1-x}{2}\right)^{s-1} dx \\
&= \frac{1}{2} \frac{1}{2^{2(s-1)}} \int_{-1}^1 (1-x^2)^{s-1} dx = 2^{1-2s} \left(2\int_0^1 (1-x^2)^{s-1} dx\right) \\
&= 2^{1-2s} B\left(\frac{1}{2}, s\right) = 2^{1-2s} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(s)}{\Gamma\left(s + \frac{1}{2}\right)}
\end{aligned}
$$

Where we used the previous result in the second last equality. If we solve this equation for $\Gamma(2s)$ using $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ we get:

$$
\Gamma(2s) = \frac{1}{\sqrt{\pi}} 2^{2s-1} \Gamma(s) \Gamma\left(s + \frac{1}{2}\right)
$$

e) Show that

$$
\begin{aligned}
\mathcal{M}\left(e^{-y^2}\right)(s) &= \frac{1}{2}\Gamma\left(\frac{s}{2}\right) && \text{for any } s \in H(0), \\
\mathcal{M}\left(\frac{e^{-y}}{1-e^{-y}}\right)(s) &= \Gamma(s)\zeta(s) && \text{for any } s \in H(1).
\end{aligned}
$$

Solution : The first identity follows from the change of variables $x = y^2$ and the domain of definition remains the same.

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Let $g(y,s) = \frac{e^{-y}}{1-e^{-y}} y^{s-1}$. For $y > 1$ we have the bound $|g(y,s)| \leq \frac{e}{e-1} e^{-y} y^{\mathrm{Re}(s)-1}$. Let $0 \leq y \leq 1$. Since $1 + y \leq e^y$ we have:

$$|g(y,s)| \leq \frac{e^{-y}}{1-\frac{1}{1+y}} y^{\mathrm{Re}(s)-1} = \frac{e^{-y}(1+y)}{y} y^{\mathrm{Re}(s)-1} \leq 2e^{-y} y^{\mathrm{Re}(s)}$$

Comparing both results we get for $y \in \mathbb{R}^+$: $|g(y,s)| \leq 2e^{-y} y^{\mathrm{Re}(s)}$. Hence, $< 1, \infty >$ is the fundamental strip for $\frac{e^{-y}}{1-e^{-y}}$.

By the second property of exercise $2b$ we have:

$$\mathcal{M}\left(\frac{e^{-y}}{1-e^{-y}}\right)(s) = \int_0^\infty \sum_{n=1}^\infty e^{-yn} y^s \frac{dy}{y} = \sum_{n=1}^\infty \mathcal{M}(e^{-yn})(s) = \sum_{n=1}^\infty n^{-s} \mathcal{M}(e^{-y})(s)$$
$$= \zeta(s)\Gamma(s)$$

4. a) Take a modular form $f \in \mathcal{M}_k(\Gamma)$ with q-expansion $f = \sum a(n)q^n$. Let χ be a character mod p, where p is a prime, and set

$$f_\chi(z) \;=\; \sum a(n)\chi(n)q^n.$$

Show that $f_\chi \in \mathcal{M}_k(\Gamma_0(p^2), \chi^2)$, i.e.

$$f_\chi(\gamma z) \;=\; \chi(d)^2 (cz+d)^k f(z).$$

Moreover, show that if $f \in \mathcal{S}_k(\Gamma)$, then $f_\chi \in \mathcal{S}_k(\Gamma_0(p^2), \chi^2)$.

Solution : We twist f_χ by the Gauss sum $G(1,\overline{\chi})$ and obtain (recall ex 1, serie 4)

$$G(1,\overline{\chi})f_\chi \;=\; \sum_{n\geq 0} \left(G(1,\overline{\chi})\chi(n)\right) a(n)q^n \;=\; \sum_{n\geq 0} G(n,\overline{\chi})a(n)q^n$$
$$= \sum_{n\geq 0} \left(\sum_{m \bmod p} \overline{\chi}(m) e^{2\pi i \frac{nm}{p}} \right) a(n)q^n \;=\; \sum_{m \bmod p} \overline{\chi}(m) f|_k \begin{pmatrix} 1 & m/p \\ & 1 \end{pmatrix}.$$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p^2)$. We will show that

$$G(1,\overline{\chi})f_\chi|_k\gamma \;=\; G(1,\overline{\chi})\chi(d)^2 f_\chi.$$

Observe that

$$\begin{pmatrix} 1 & m/p \\ & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; \begin{pmatrix} a + \frac{m}{p}c & b + \frac{dm}{p} \\ c & d \end{pmatrix}.$$

This product might not be in $\Gamma_0(p^2)$ since the upper right entry is not necessarily integral. However,

$$\begin{pmatrix} a + \frac{m}{p}c & b + \frac{dm}{p} \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -d^2\frac{m}{p} \\ & 1 \end{pmatrix} \;=\; \begin{pmatrix} a + \frac{m}{p}c & b - \frac{bcdm}{p} - \frac{cd^2m^2}{p^2} \\ c & d - \frac{m}{p}d^2c \end{pmatrix} \;=:\; \gamma' \in \Gamma_0(p^2).$$

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It now follows that

$$G(1,\overline{\chi})f_\chi|_k\gamma \;=\; \sum_{m \bmod p} \overline{\chi}(m) f|_k\gamma' \begin{pmatrix} 1 & d^2\frac{m}{p} \\ & 1 \end{pmatrix} \;=\; \chi(d)^2 \sum_{k \bmod p} \overline{\chi}(k)(f|_k\gamma')|_k \begin{pmatrix} 1 & \frac{k}{p} \\ & 1 \end{pmatrix}$$
$$= \chi(d)^2 G(1,\overline{\chi})(f|_k\gamma')_\chi \;=\; \chi(d)^2 G(1,\overline{\chi}) f_\chi.$$

Finally, one can read immediately from the q-expansion for f_χ that it is holomorphic at infinity (resp. vanishes at infinity) if f does.

b) Given N a positive integer, let $\omega_N := \begin{pmatrix} & -1 \\ N & \end{pmatrix}$. Show that ω_N normalizes $\Gamma_0(N)$ and that if $f \in \mathcal{M}_k(\Gamma_0(N))$, then

$$f|_{\omega_N} \;=\; N^{-k/2} z^{-k/2} f\left(\frac{-1}{Nz}\right)$$

is also in $\mathcal{M}_k(\Gamma_0(N))$.

Solution : The direct computation

$$\begin{pmatrix} & 1 \\ -N & \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} & -1 \\ N & \end{pmatrix} \;=\; \begin{pmatrix} dN & -c \\ -bN^2 & aN \end{pmatrix} \;=:\; \gamma',$$

once you divide both sides by N proves that ω_N normalizes $\Gamma_0(N)$.

Let $\gamma \in \Gamma_0(N)$, then

$$f|_{\omega_N}\gamma \;=\; f|\omega_N\gamma \;=\; f|\gamma'\omega_N \;=\; f|_{\omega_N}.$$

c) Let $f \in \mathcal{S}_k(\Gamma)$, and let χ be a character mod p. Show that $f_\chi|_{\omega_{p^2}} = \frac{\tau(\chi)^2}{p} f_{\overline{\chi}}$, where $\tau(\chi) = G(1,\chi)$ denotes the Gauss sum.

Solution : We know from last exercise sheet that $|G(1,\chi)| = p$. We show the therefore equivalent statement

$$|G(1,\chi)| f_\chi|_{\omega_p^2} \;=\; G(1,\chi) f_{\overline{\chi}}.$$

By the same computation that in part **a)**, we can show that

$$G(1,\chi) f_{\overline{\chi}} \;=\; \sum_{m \bmod p} \chi(m) f|_k \begin{pmatrix} 1 & m/p \\ & 1 \end{pmatrix} \;=\; \overline{G(1,\chi)} f_\chi.$$

It follows from the direct computation

$$\begin{pmatrix} 1 & m/p \\ & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ p^2 & \end{pmatrix}\begin{pmatrix} 1 & -m/p \\ & 1 \end{pmatrix} \;=\; \begin{pmatrix} mp & -1-m^2 \\ p^2 & -mp \end{pmatrix} \;=:\; \gamma \in \Gamma$$

that

$$f|_k \begin{pmatrix} 1 & m/p \\ & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ p^2 & \end{pmatrix} \;=\; f|_k\gamma \begin{pmatrix} 1 & m/p \\ & 1 \end{pmatrix} \;=\; f|_k \begin{pmatrix} 1 & m/p \\ & 1 \end{pmatrix}$$

and we conclude.

5. Let again $f \in \mathcal{S}_k(\Gamma)$, and let χ be a Dirichlet character mod p. Set

$$L(f,\chi,s) \;=\; \sum_{n\geq 1} \frac{a(n)\chi(n)}{n^s} \quad \text{and} \quad \Lambda(f,\chi,s) \;=\; \left(\frac{p}{2\pi}\right)^s \Gamma(s)L(f,\chi,s).$$

Prove the functional equation $\Lambda(f,\chi,s) = i^k \frac{\tau(\chi)^2}{p}\Lambda(f,k-s,\overline{\chi})$.

Solution : Consider the function $\widetilde{f}_\chi(y) := f_\chi(iy)$. Then

$$\mathcal{M}(\widetilde{f}_\chi)(s) \;=\; \int_0^\infty f_\chi(iy)y^s\frac{dy}{y} \;=\; (2\pi)^{-s}\Gamma(s)L(f,\chi,s).$$

On the other hand, we obtain from the change of variables $y = 1/(p^2u)$ that

$$\mathcal{M}(\widetilde{f}_\chi)(s) \;=\; p^{-2s}\int_0^\infty f_\chi\left(\frac{i}{p^2u}\right)u^{-s}\frac{du}{u}.$$

From Ex. 4, part c), we know that

$$f_\chi|_{\omega_p^2}(iu) \;=\; (ipu)^{-k}f_\chi\left(\frac{i}{p^2u}\right) \;=\; \frac{\tau(\chi)^2}{p}f_{\overline{\chi}}(iu)$$

hence

$$\mathcal{M}(\widetilde{f_\chi})(s) \;=\; p^{-2s}(ip)^k\frac{\tau(\chi)^2}{p}\int_0^\infty f_{\overline{\chi}}(iu)u^{k-s}\frac{du}{u} \;=\; p^{-2s}(ip)^k\frac{\tau(\chi)^2}{p}\mathcal{M}(\widetilde{f_{\overline{\chi}}})(k-s)$$

that is,

$$(2\pi)^{-s}\Gamma(s)L(f,\chi,s) \;=\; p^{-2s}(ip)^k\frac{\tau(\chi)^2}{p}(2\pi)^{-(k-s)}\Gamma(k-s)L(f,\overline{\chi},k-s).$$

This equality is equivalent to $\Lambda(f,\chi,s) = i^k\frac{\tau(\chi)^2}{p}\Lambda(f,\overline{\chi},k-s)$.