Solutions 7

1. Let $z \in \mathbb{H}$ and consider the Θ-function defined by

$$\Theta_z(t) \;=\; \sum_{m,n\in\mathbb{Z}} e^{-\pi t \frac{|mz+n|^2}{y}}$$

for all $t > 0$.

a) Show that Θ_z satisfies the functional equation $\Theta_z(t) \;=\; \frac{1}{t}\Theta_z\left(\frac{1}{t}\right)$.

Proof:

$$\begin{aligned}
\Theta_z(t) &\overset{\text{def}}{=} \sum_{m,n\in\mathbb{Z}} e^{-\pi\frac{t}{y}\left((mx+n)^2+(my)^2\right)} \\
&= \sum_{m\in\mathbb{Z}} e^{-\pi\frac{t}{y}(my)^2} \sum_{n\in\mathbb{Z}} e^{-\pi\frac{t}{y}(mx+n)^2} \\
&\overset{(*)}{=} \sum_{m\in\mathbb{Z}} e^{-\pi\frac{t}{y}(my)^2} \sum_{n\in\mathbb{Z}} e^{2\pi imxn}\sqrt{\frac{y}{t}}\,e^{-\pi\frac{y}{t}n^2} \\
&= \sqrt{\frac{y}{t}} \sum_{m,n\in\mathbb{Z}} e^{-\pi tym^2} e^{2\pi imxn} e^{-\pi\frac{y}{t}n^2} \\
&= \sqrt{\frac{y}{t}} \sum_{n\in\mathbb{Z}} e^{-\pi\left(\frac{x^2+y^2}{ty}\right)n^2} \sum_{m\in\mathbb{Z}} e^{-\pi ty\left(m+\frac{inx}{ty}\right)^2} \\
&\overset{(*)}{=} \sqrt{\frac{y}{t}} \sum_{n\in\mathbb{Z}} e^{-\pi\left(\frac{x^2+y^2}{ty}\right)n^2} \sum_{m\in\mathbb{Z}} \frac{1}{\sqrt{ty}} e^{-\pi\frac{m^2}{ty}}\, e^{-\frac{\pi}{ty}(2mnx)} \\
&= \frac{1}{t}\sum_{m,n\in\mathbb{Z}} e^{-\frac{\pi}{ty}\left(n^2(x^2+y^2)+m^2+2mnx\right)} \overset{\text{def}}{=} \frac{1}{t}\Theta_z\left(\frac{1}{t}\right)
\end{aligned}$$

where $(*)$ indicates that we applied the Poisson summation formula.

For all $s \in \langle 1, \infty\rangle$, let

$$E(z,s) \;=\; \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}(\gamma z)^s \;=\; \sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} \frac{y^s}{|cz+d|^{2s}}$$

and

$$E^*(z,s) \;=\; \pi^{-s}\Gamma(s)2\zeta(2s)E(z,s) \;=\; \pi^{-s}\Gamma(s) \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{y^s}{|mz+n|^{2s}}.$$

b) Check that $E(\gamma z, s) = E(z,s)$ for all $\gamma \in \Gamma$ and show that

$$E^*(z,s) \;=\; \int_0^\infty \left(\Theta_z(t)-1\right) t^s \frac{dt}{t}.$$

Solution: Let $\gamma' \in \Gamma$, then the collection of elements $\gamma\gamma'$ where γ runs through a system of coset representatives for $\Gamma_\infty\backslash\Gamma$ is also a system of coset representatives for that quotient, hence $E(\gamma' z, s) = E(z,s)$ and this holds for all $\gamma' \in \Gamma$.

Observe that

$$\Theta_z(t) - 1 \;=\; \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} e^{-\pi\frac{t}{y}|mz+n|^2}$$

and

$$\begin{aligned}\int_{\mathbb{R}_{>0}} \left(\Theta_z(t)-1\right) t^s \frac{dt}{t} \quad &= \quad \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \mathcal{M}\left(t \mapsto e^{-\pi\frac{t}{y}|mz+n|^2}\right)(s)\\ &\overset{(**)}{=} \quad \mathcal{M}(t\mapsto e^{-t})(s) \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \left(\frac{\pi|mz+n|^2}{y}\right)^{-s} \;=\; E^*(z,s)\end{aligned}$$

where in $(**)$ we applied the second transformation property of Mellin transforms that we proved in exercise 2b of problem set 5.

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c) Show that $E^*(z,s)$ has a meromorphic continuation to the whole complex s-plane with single poles at $s=0$ and $s=1$ with residues -1 and 1 respectively. Finally, prove the functional equation $E^*(z,1-s) = E^*(z,s)$.

Solution : We decompose the integral representation of $E^*(z,s)$ into the sum of the two integrals

$$E^*(z,s) \;=\; \int_0^1 \left(\Theta_z(t)-1\right) t^s \frac{dt}{t} + \int_1^\infty \left(\Theta_z(t)-1\right) t^s \frac{dt}{t}.$$

Observe that the first integral is analytic for $s \in \langle 1, \infty\rangle$ while the second integral is entire in that fundamental strip. For the first integral, we have

$$\begin{aligned}\int_0^1 (\Theta_z(t)-1)\, t^s \frac{dt}{t} &= \int_1^\infty \left(\Theta_z\left(\frac{1}{t}\right)-1\right) t^{-s}\frac{dt}{t} \\ &\overset{a)}{=} \int_1^\infty \left(\Theta_z(t)-\frac{1}{t}\right) t^{1-s}\frac{dt}{t} \\ &= \int_1^\infty (\Theta_z(t)-1)\, t^{1-s}\frac{dt}{t} + \int_1^\infty \left(1-\frac{1}{t}\right) t^{1-s}\frac{dt}{t} \\ &= \int_1^\infty (\Theta_z(t)-1)\, t^{1-s}\frac{dt}{t} + \frac{1}{s(1-s)}.\end{aligned}$$

Again, the first integral is entire in s and it follows that $E^*(z,s)$ has a meromorphic continuation with poles at $s=0,1$ and $E^*(z,1-s)=E^*(z,s)$.

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2. Let $\varphi:\mathbb{H}\to\mathbb{C}$ be an analytic function such that $\varphi(\gamma z)=\varphi(z)$ for all $\gamma\in\Gamma$ and $\varphi(z)=O\left(y^{-C}\right)$ as $y\to\infty$ for all $C>0$. Such a function has a Fourier expansion of the form $\varphi(z)=\sum_{n\in\mathbb{Z}}\varphi_n(y)e^{2\pi inx}$ where $\varphi_n(y)=\int_0^1\varphi(x+iy)e^{-2\pi inx}dx$. Set

$$\Lambda_\varphi(s) \;=\; \pi^{-s}\Gamma(s)2\zeta(2s)\mathcal{M}(\varphi_0)(s-1)$$

for all $s\in\langle 1,\infty\rangle$.

a) Show that $\mathcal{M}(\varphi_0)(s)$ is indeed well-defined on the fundamental strip $\langle 0,\infty\rangle$ and that it is bounded in every vertical strip strictly contained in $\langle 0,\infty\rangle$.

Proof: First of all, we show that a Γ-invariant function φ that decays rapidly in the cusp as described above, is a bounded function. By invariance, φ can be seen as a function on the closure $\overline{\mathcal{F}}$ of the standard fundamental domain for Γ. Now,

$$\overline{\mathcal{F}} \;=\; \left(\overline{\mathcal{F}}\cap\{y\le C\}\right)\cup\left(\overline{\mathcal{F}}\cap\{y>C\}\right).$$

For any positive constant C, the first component defines a compact region on which φ is then necessarily bounded. We can choose C sufficiently large so that φ, which is rapidly decaying as $y\to\infty$, is also bounded on the second component.

Then $\varphi_0(y)=\int_0^1\varphi(x+iy)e^{-2\pi inx}dx$ is also bounded and rapidly decaying in the cusp.

Let $s\in\langle a,b\rangle$, a vertical strip strictly contained in the fundamental strip $\langle 0,\infty\rangle$. Then

$$|\mathcal{M}(\varphi_0)(s)| \;\le\; \int_0^1|\varphi_0(y)|y^a\frac{dy}{y}+\int_1^M|\varphi_0(y)|y^b\frac{dy}{y}+\int_M^\infty|\varphi_0(y)|y^b\frac{dy}{y} \;<\;\infty$$

where we chose M such that $|\varphi_0(y)|\ll y^{-b-1}$ whenever $y\ge M$.

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b) Check that Λ_φ has the following integral representation

$$\Lambda_\varphi(s) \;=\; \langle \varphi, \overline{E^*(\cdot,s)}\rangle \;=\; \int_{\mathcal{F}} \varphi(z)E^*(z,s)d\mu(z)$$

where $\mathcal{F}$ denotes a fundamental domain for Γ.

Solution: Note that it suffices to show that $\mathcal{M}(\varphi_0)(s-1) = \langle \varphi, \overline{E(\cdot,s)}\rangle$. And indeed,

$$\mathcal{M}(\varphi_0)(s-1) \overset{\text{def}}{=} \int_0^\infty \varphi_0(y)y^s\frac{dy}{y^2} \overset{\text{def}}{=} \int_0^\infty\int_0^1 \varphi(x+iy)dx\, y^s\frac{dy}{y^2} \;=\; \int_{\mathcal{F}_\infty} \varphi(z)y^s d\mu(z),$$

where $\mathcal{F}_\infty = \{z\in\mathbb{H} : x\in[0,1]\}$. One can choose a collection of representatives (α_j) such that $\mathcal{F}_\infty = \bigcup_{\alpha\in\Gamma_\infty\backslash\Gamma}\alpha^{-1}\mathcal{F}$. Then

$$\begin{aligned}\int_{\mathcal{F}_\infty}\varphi(z)y^s d\mu(z) &= \sum_{\alpha\in\Gamma_\infty\backslash\Gamma}\int_{\alpha^{-1}\mathcal{F}}\varphi(z)y^s d\mu(z)\\ &= \sum_{\alpha\in\Gamma_\infty\backslash\Gamma}\int_{\mathcal{F}}\varphi(z)\,\mathrm{Im}(\alpha z)^s d\mu(z) \;=\; \int_{\mathcal{F}}\varphi(z)E(z,s)d\mu(z).\end{aligned}$$

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c) Prove that Λ_φ has a meromorphic continuation to the whole complex plane with simple poles at $s=0$ and $s=1$ with residues $\mp\int_{\mathcal{F}}\varphi(z)d\mu(z)$. It is bounded in any vertical strip (that does not contain a pole) and satisfies the functional equation

$$\Lambda_\varphi(s) \;=\; \Lambda_\varphi(1-s).$$

N.B. This is the simplest case of the Rankin–Selberg method.

Proof: By the assumptions on φ and ex 1c), Λ_φ is analytic for $s\neq 0,1$, where it admits simple poles, coming from the simple poles of $E^*(z,s)$. We can conclude that it is moreover bounded on vertical strips from part a) of this exercise. Finally,

$$\operatorname*{Res}_{s=1}(\Lambda_\varphi) \;=\; \int_{\mathcal{F}}\operatorname*{Res}_{s=1}E^*(z,s)\varphi(z)d\mu(z) \;=\; \int_{\mathcal{F}}\varphi(z)d\mu(z),$$

and by the functional equation from 1c) $\mathrm{Res}(\Lambda_\varphi)(1) = -\mathrm{Res}(\Lambda_\varphi)(0)$.

Let $f=\sum a_n q^n\in\mathcal{S}_k(\Gamma)$ and $g=\sum b_n q^n\in\mathcal{M}_k(\Gamma)$ and set $\phi = f\overline{g}y^k$. We define

$$\begin{aligned} L(f\times g,s) &= 2\zeta(2s-2k+2)\sum_{n\geq 1}a_n\overline{b_n}n^{-s},\\ \Lambda(f\times g,s) &= \pi^{k-1}(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+1)L(f\times g,s).\end{aligned}$$

For simplicity, we will assume that $b_n=\overline{b_n}$ for all n.

The L-series $L(f\times g,s)$ is called the Rankin–Selberg convolution of f and g.

d) Check that ϕ satisfies the same properties as the function φ at the beginning of the exercise. Show that for all $s \in \langle 0, \infty\rangle$

$$\mathcal{M}(\phi_0)(s) \;=\; (4\pi)^{-(s+k)}\Gamma(s+k)\sum_{n\geq 1} a_n\overline{b_n}n^{-(s+k)}.$$

Solution: Clearly φ is analytic. Since f is a cusp form, $f(z) = O(y^{-C})$ as $y \to \infty$ for any $C > 0$. In particular, let $f(z) = O(y^{-C-k})$ also holds. The function g is a modular form and therefore $g(z) = O(1)$, hence $\phi(z) = O(y^{-C})$ as $y \to \infty$. Finally,

$$\phi(\gamma z) \;=\; \frac{j(\gamma,z)^k\overline{j(\gamma,z)^k}}{|j(\gamma,z)|^{2k}}\phi(z) \;=\; \phi(z)$$

for all $\gamma \in \Gamma$ and $z \in \mathbb{H}$.

We compute

$$\phi_0(y) \;\overset{\text{def}}{=}\; \int_0^1 \phi(x+iy)dy \;=\; \sum_{m\geq 1}\sum_{n\geq 0} a_m b_n y^k e^{-2\pi(m+n)}\int_0^1 e^{2\pi i(m-n)x}dx \;=\; \sum_{n\geq 1} a_n b_n y^k e^{-4\pi ny}.$$

By Hecke's estimate, $|a_n| = O(n^{\frac{k}{2}})$ and $|b_n| = O(n^k)$. Hence

$$\begin{aligned}
\mathcal{M}(\phi_0)(s) \quad &= \quad \sum_{n\geq 1} a_n b_n \int_{\mathbb{R}_{>0}} e^{-4\pi ny}y^{s+k}\frac{dy}{y} \\
&= \quad \sum_{n\geq 1} a_n b_n \left((4\pi n)^{-(s+k)}\Gamma(s+k)\right) \\
&= \quad (4\pi)^{-(s+k)}\Gamma(s+k)\sum_{n\geq 1}\frac{a_n b_n}{n^{s+k}}.
\end{aligned}$$

for all $s \in \langle k/2+2, \infty\rangle$. We finally show that this actually holds on the strip $\langle 0, \infty\rangle$.

We assume that the sum doesn't converge absolutely anymore for $\mathrm{Re}(s) \leq \sigma$ and choose σ maximally. By contradiction we assume that $\sigma > 0$. Note that the equality still holds for $s \in <\sigma + \frac{1}{N}, \infty>$ for all $N > 0$. Also note that $\mathrm{Im}(s)$ doesn't matter for the absolute convergence. Since the sum doesn't converge absolutely for $\mathrm{Re}(s) \leq \sigma$ we can (wlog) assume that the following series diverges as $N \to \infty$:

$$S_N := \sum_{n\geq 1} a_n\overline{b_n}n^{-(\sigma+\frac{1}{N}+k)}$$

But on the other hand we have:

$$S_N = (4\pi)^{\sigma+\frac{1}{N}+k}\Gamma\left(\sigma+\frac{1}{N}+k\right)\mathcal{M}(\phi_0(y))(\sigma+\frac{1}{N})$$

Which converges for $N \to \infty$ to $(4\pi)^{\sigma+k}\Gamma(\sigma+k)\mathcal{M}(\phi_0(y))(\sigma) \in \mathbb{C}$ (as shown before). This gives a contradiction. Hence the formula holds for $s \in < 0, \infty >$.

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e) Prove that $\Lambda(f \times g, s)$ has a meromorphic continuation to the whole complex plane with simple poles at $s = k$ and $s = k-1$ with residues $\pm\langle f, g\rangle$. It is bounded in any vertical strip (that does not contain a pole) and satisfies the functional equation

$$\Lambda(f \times g, s) \;=\; \Lambda(f \times g, 2k-1-s).$$

Hint: Show first that $\Lambda_\phi(s) = \Lambda(f \times g, s+k-1)$.

Proof: By the definitions,

$$\Lambda(f \times g, s+k-1) \;=\; \frac{\pi^{k-1}}{4^{s+k-1}} 2\zeta(2s)\Gamma(s)\Gamma(s+k-1)\sum_{n\geq 1}\frac{a_n b_n}{n^{s+k-1}}$$

and

$$\begin{aligned}\Lambda_\phi(s) &\overset{\text{def}}{=} \pi^{-s}\Gamma(s)2\zeta(2s)\mathcal{M}(\phi_0)(s-1)\\ &\overset{2d}{=} \frac{\pi^{-s}}{(4\pi)^{s+k-1}}\frac{4^{s+k-1}}{\pi^{k-1}}\Lambda(f \times g, s+k-1) \;=\; \Lambda(f \times g, s+k-1)\end{aligned}$$

for all $s \in \langle 1, \infty\rangle$. Now we can derive the statement from the meromorphic continuation of Λ_ϕ and its properties established in exercise 2c.

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3. The MacDonald–Bessel function is given by

$$K_s(y) \;=\; \frac{1}{2}\int_0^\infty e^{-\frac{y}{2}\left(t+\frac{1}{t}\right)} t^s \frac{dt}{t}$$

for all $y > 0$, $s \in \mathbb{C}$. It is entire as a function in s and decays rapidly as $y \to \infty$. Moreover, one can show by a change of variable that $K_s(y) = K_{-s}(y)$.

a) Set

$$I_s(a) \;=\; \int_{\mathbb{R}} \frac{e^{iau}}{(u^2+1)^s} du$$

for all $a \in \mathbb{R}$, $s \in \langle 1/2, \infty\rangle$. Prove that

$$\Gamma(s)I_s(a) \;=\; \begin{cases}\sqrt{\pi}\,\Gamma(s-1/2) & a = 0\\ 2\sqrt{\pi}\left|\frac{a}{2}\right|^{s-1/2} K_{s-1/2}(|a|) & a \neq 0.\end{cases}$$

Proof:

$$\begin{aligned}\Gamma(s)I_s(a) &= \int_0^\infty\int_{-\infty}^\infty e^{-t}t^{s-1}\frac{e^{iau}}{(u^2+1)^s}dudt\\ &= \int_0^\infty\int_{-\infty}^\infty e^{-(u^2+1)t}t^{s-1}e^{iau}dudt\\ &= \int_0^\infty\left(\int_{-\infty}^\infty e^{-(u^2+1)t}e^{iau}du\right)t^s\frac{dt}{t}.\end{aligned}$$

If $a = 0$, then the inner integral is equal to $\sqrt{\pi}e^{-t}t^{-1/2}$, and

$$\Gamma(s)I_s(0) \;=\; \sqrt{\pi}\int_0^\infty e^{-t}t^{s-1/2}\frac{dt}{t} \;=\; \sqrt{\pi}\,\Gamma(s-1/2).$$

Otherwise, it is equal to

$$\begin{aligned} e^{-t}\int_{\mathbb{R}} e^{iau}e^{-tu^2}du &= 2\pi e^{-t}\int_{\mathbb{R}} e^{-4\pi^2u^2t}e^{-2\pi iau}du \\ &= 2\pi e^{-t}\left(\frac{1}{2\sqrt{\pi t}}e^{-\frac{\pi a^2}{4\pi t}}\right) \;=\; \sqrt{\pi}\frac{e^{-t}}{\sqrt{t}}e^{-a^2/(4t)} \\ &= \sqrt{\frac{\pi}{t}}\,e^{-\frac{|a|}{2}\left(\frac{|a|}{2}t+\frac{1}{\frac{|a|}{2}t}\right)}. \end{aligned}$$

Hence

$$\begin{aligned} \Gamma(s)I_s(a) &= \sqrt{\pi}\mathcal{M}\left(t\mapsto e^{-\frac{|a|}{2}\left(\frac{|a|}{2}t+\frac{1}{\frac{|a|}{2}t}\right)}\right)(s-1/2) \\ &= \sqrt{\pi}\left(\frac{|a|}{2}\right)^{s-1/2}\mathcal{M}\left(e^{-\frac{|a|}{2}\left(t+\frac{1}{t}\right)}\right)(s-1/2) \\ &= \sqrt{\pi}\left(\frac{|a|}{2}\right)^{s-1/2}K_{s-1/2}(|a|). \end{aligned}$$

Let $s \in \langle 1,\infty\rangle$ and consider the Fourier expansion $E^*(z,s) = \sum_{n\in\mathbb{Z}} a_n(y,s)e^{2\pi inx}$ with coefficients

$$\begin{aligned} a_0(y,s) &= 2\Lambda(2s)y^s + 2\Lambda(2s-1)y^{1-s} \\ a_n(y,s) &= 4\sqrt{y}|n|^{s-1/2}\sigma_{1-2s}(|n|)K_{s-1/2}(2\pi|n|y) \end{aligned}$$

where $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

b) Prove that each coefficient $a_n(y,s)$, $n \neq 0$, has an analytical continuation to an entire function and satisfies the functional equation

$$a_n(y,s) \;=\; a_n(y,1-s).$$

Proof: Each $a_n(y,s)$, $n \neq 0$, is entire, since $K_s(y)$ and $\sigma_s(|n|)$ are entire. To show the functional equation, we first compute

$$\begin{aligned} |n|^{1/2-s}\sigma_{2s-1}(|n|) &= |n|^{1/2-s}\sum_{d||n|} d^{2s-1} \\ &= |n|^{s-1/2}\sum_{d||n|}\frac{d^{2s-1}}{|n|^{1-2s}} \\ &= |n|^{s-1/2}\sum_{d||n|} d^{1-2s} \;=\; |n|^{s-1/2}\sigma_{1-2s}(|n|). \end{aligned}$$

Then

$$\begin{aligned} a_n(y,s) &\overset{\text{def}}{=} 4\sqrt{y}\left(|n|^{s-1/2}\sigma_{1-2s}(|n|)\right)K_{s-1/2}(2\pi|n|y) \\ &= 4\sqrt{y}\left(|n|^{1/2-s}\sigma_{2s-1}(|n|)\right)K_{1/2-s}(2\pi|n|y) \;=\; a_n(y,1-s). \end{aligned}$$

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c) Show that $\Lambda(s)$ has a meromorphic continuation to the whole complex plane with simple poles at $s = 0, 1$ with residues ∓ 1, and that it satisfies the functional equation

$$\Lambda(s) \;=\; \Lambda(1-s).$$

Proof: It follows from exercises 3b and 1c that the constant term $a_0(y, s)$ has a meromorphic continuation to the whole complex plane, with poles at $s = 0, 1$, residues ∓ 1 and functional equation $a_0(y, s) = a_0(y, 1 - s)$.

If we observe that

$$a_0\left(y, \frac{s}{2}\right)\frac{y^{s/2-1}}{2} \;=\; \Lambda(s)y^{s-1} + \Lambda(s-1),$$

we can express $\Lambda(s)$ as

$$\Lambda(s) \;=\; \Lambda(s)\left(\frac{y_1^{s-1} - y_2^{s-1}}{y_1^{s-1} - y_2^{s-1}}\right) \;=\; \frac{1}{2}\frac{a_0(y_1, s/2)y_1^{s/2-1} - a_0(y_2, s/2)y_2^{s/2-1}}{y_1^{s-1} - y_2^{s-1}}$$

where $y_1, y_2 \in \mathbb{R}_{>0}$ are distinct. Now $\Lambda(s)$ has a meromorphic continuation, and is analytic outside of $s = 0, 1, 2$. Moreover, from the functional equation for a_0, one has

$$\Lambda(s)y^{s/2} + \Lambda(s-1)y^{1-s/2} \;=\; \Lambda(2-s)y^{1-s/2} + \Lambda(1-s)y^{s}$$

or equivalently,

$$\Lambda(s) - \Lambda(1-s) \;=\; \left(\Lambda(2-s) - \Lambda(s-1)\right)y^{1-s}.$$

The latter equality can only be true if $\Lambda(s) = \Lambda(1 - s)$.

To compute the residue at $s = 0$, we choose $y_1 = y$ and $y_2 = -y$,

$$\operatorname*{Res}_{s=0} \Lambda(s) \;=\; \frac{y}{4}\left(\frac{-4}{y}\right) \;=\; -1.$$

It follows from then functional equation that the residue at $s = 1$ is then $= +1$. Finally, choosing for $\Lambda(s)$ $y_1 = y_2 = y$, we note that $s = 2$ is a removable singularity.

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4. **a)** Let $w \in \mathbb{C}$. Show that:

$$\mathcal{M}\left(K_w(y)\right)(s) = 2^{s-2}\Gamma\left(\frac{s+w}{2}\right)\Gamma\left(\frac{s-w}{2}\right)$$

Proof:

$$\begin{aligned}
2^{s-2}\Gamma\left(\frac{s+w}{2}\right)\Gamma\left(\frac{s-w}{2}\right) &= 2^{s-2}\int_0^\infty\int_0^\infty e^{-(y+z)}y^{\frac{s+w}{2}}z^{\frac{s-w}{2}}\frac{dy}{y}\frac{dz}{z} \\
\left(\text{setting } y = t^2 z,\ \frac{dy}{y} = 2\frac{dt}{t}\right) &= 2^s\frac{1}{2}\int_0^\infty\int_0^\infty e^{-(t^2z+z)}t^{s+w}z^s\frac{dt}{t}\frac{dz}{z} \\
&= \frac{1}{2}\int_0^\infty\int_0^\infty e^{-\frac{2tz}{2}\left(t+\frac{1}{t}\right)}(2tz)^s t^w\frac{dt}{t}\frac{dz}{z} \\
\left(\text{setting } y = 2tz,\ \frac{dz}{z} = \frac{dy}{y}\right) &= \int_0^\infty\left(\frac{1}{2}\int_0^\infty e^{-\frac{y}{2}\left(t+\frac{1}{t}\right)}t^w\frac{dt}{t}\right)y^s\frac{dy}{y} \\
&= \int_0^\infty K_w(y)y^s\frac{dy}{y} = \mathcal{M}\left(K_w(y)\right)(s)
\end{aligned}$$

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b) Use task $3b, c)$ and $4a)$ to show that:

$$\Lambda(E^*(\cdot,w),s) := \mathcal{M}\left(E^*(iy,w)) - a_0(y,w)\right)(s) = 2\Lambda(s+w)\Lambda(s+1-w) = 2\Lambda(w+s)\Lambda(w-s)$$

Hint: Show and use the following fact:

$$\sum_{n=1}^\infty \sigma_w(n)n^{-s} = \zeta(s)\zeta(s-w)$$

Proof:

$$\begin{aligned}
\sum_{n=1}^\infty \sigma_w(n)n^{-s} &= \sum_{n=1}^\infty\sum_{d|n} d^w n^{-s} = \sum_{n=1}^\infty\sum_{\substack{a,b\in\mathbb{Z}_{>0}\\ ab=n}} a^w(ab)^{-s} = \sum_{a,b\in\mathbb{Z}_{>0}} a^{-(s-w)}b^{-s} \\
&= \sum_{a=1}^\infty a^{-s(s-w)}\sum_{b=1}^\infty b^{-s} = \zeta(s)\zeta(s-w)
\end{aligned}$$

First note that

$$E^*(iy,w) - a_0(y,w) = \sum_{n\neq 0} a_n(y,w) = 8y^{\frac{1}{2}}\sum_{n=1}^\infty n^{w-\frac{1}{2}}\sigma_{1-2w}(n)K_{w-\frac{1}{2}}(2\pi n y)$$

Since the Mellin transformation is linear we only need to compute

$$\begin{aligned}
\mathcal{M}\left(y^{\frac{1}{2}}K_{w-\frac{1}{2}}(2\pi ny)\right)(s) &= \mathcal{M}\left(K_{w-\frac{1}{2}}(2\pi ny)\right)\left(s+\frac{1}{2}\right) \\
&= (2\pi n)^{-\left(s+\frac{1}{2}\right)}\mathcal{M}\left(K_{w-\frac{1}{2}}(y)\right)\left(s+\frac{1}{2}\right) \\
&= (2\pi n)^{-\left(s+\frac{1}{2}\right)}\left(2^{s-2+\frac{1}{2}}\Gamma\left(\frac{s+\frac{1}{2}+w-\frac{1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{1}{2}-w+\frac{1}{2}}{2}\right)\right) \\
&= \frac{1}{4}\pi^{-\left(s+\frac{1}{2}\right)}n^{-\left(s+\frac{1}{2}\right)}\Gamma\left(\frac{s+w}{2}\right)\Gamma\left(\frac{s+1-w}{2}\right)
\end{aligned}$$

Finally,

$$\begin{aligned}
\Lambda(E^*(\cdot,w),s) &= \mathcal{M}\left(E^*(iy,w)-a_0(y,w)\right)(s) = 8\sum_{n=1}^{\infty} n^{w-\frac{1}{2}}\sigma_{1-2w}(n)\mathcal{M}\left(y^{\frac{1}{2}}K_{w-\frac{1}{2}}(2\pi ny)\right)(s) \\
&= 8\sum_{n=1}^{\infty} n^{w-\frac{1}{2}}\sigma_{1-2w}(n)\left(\frac{1}{4}\pi^{-\left(s+\frac{1}{2}\right)}n^{-\left(s+\frac{1}{2}\right)}\Gamma\left(\frac{s+w}{2}\right)\Gamma\left(\frac{s+1-w}{2}\right)\right) \\
&= 2\pi^{-\left(s+\frac{1}{2}\right)}\Gamma\left(\frac{s+w}{2}\right)\Gamma\left(\frac{s+1-w}{2}\right)\sum_{n=1}^{\infty}\sigma_{1-2w}(n)n^{-(s+1-w)} \\
&= 2\pi^{-\left(s+\frac{1}{2}\right)}\Gamma\left(\frac{s+w}{2}\right)\Gamma\left(\frac{s+1-w}{2}\right)\zeta(s+1-w-1+2w)\zeta(s+1-w) \\
&= 2\left(\pi^{-\frac{s+w}{2}}\Gamma\left(\frac{s+w}{2}\right)\zeta(s+w)\right)\left(\pi^{-\frac{s+1-w}{2}}\Gamma\left(\frac{s+1-w}{2}\right)\zeta(s+1-w)\right) \\
&= 2\Lambda(s+w)\Lambda(s+1-w) = 2\Lambda(w+s)\Lambda(w-s)
\end{aligned}$$
