Solution sheet 4

1. Let N be any positive integer and χ be any non-tivial Dirichlet character modulo N. Let a be any integer. Then the sum

$$G(a, \chi) := \sum_{1 \leq m \leq N} \chi(m)\xi_N^{am},$$

where $\xi_N := e^{\frac{2\pi i}{N}}$, is called the *Gauss sum* associated with χ and a.

Show the following statements.

a) If $\gcd(a, N) = 1$, then $G(a, \chi) = \overline{\chi}(a)G(1, \chi)$.

Proof: If $\gcd(a, N) = 1$, then multiplication by a permutes the elements of $\mathbb{Z}/N\mathbb{Z}$. In this case we therefore we have

$$\chi(a) \sum_{m \in \mathbb{Z}/N\mathbb{Z}} \chi(m)\xi_N^{am} = \sum_{m \in \mathbb{Z}/N\mathbb{Z}} \chi(am)\xi_N^{am} = \sum_{m \in \mathbb{Z}/N\mathbb{Z}} \chi(m)\xi_N^{m}$$
$$= G(1, \chi).$$

Using moreover that $\chi(a)^{-1} = \overline{\chi}(a)$ in this case, the desired equality follows.

b) We have $G(a, \chi) = \overline{\chi}(a)G(1, \chi)$ for any integer a if and only if we have $G(a, \chi) = 0$ for any integer a with $\gcd(a, N) > 1$.

Proof: Suppose first that $G(a, \chi) = \overline{\chi}(a)G(1, \chi)$ for any integer a. By definition of $\overline{\chi}$, if a is any integer with $\gcd(a, N) > 1$, then $\overline{\chi}(a) = 0$. In particular, $G(a, \chi) = \overline{\chi}(a)G(1, \chi) = 0$ if $\gcd(a, N) > 1$.

Conversely, suppose $G(a, \chi) = 0$ for any integer a with $\gcd(a, N) > 1$. For any integer a with $\gcd(a, N) > 1$ we then have $G(a, \chi) = 0 = \overline{\chi}(a)G(1, x)$. By Part a) we moreover have for any integer a with $\gcd(a, N) = 1$ that $G(a, \chi) = \overline{\chi}(a)G(1, \chi)$. Hence the last equality holds for any integer a.

c) If $G(a, \chi) = \overline{\chi}(a)G(1, \chi)$ for any integer a, then

$$|G(1, \chi)|^2 = N. \tag{1}$$

In particular, (c)) holds if N is prime.

Proof: If $G(a,\chi) = \overline{\chi}(a)G(1,\chi)$ for any integer a, then

$$\begin{aligned} |G(1,\chi)|^2 &= G(1,\chi)\overline{G(1,\chi)} = G(1,\chi) \sum_{m\in\mathbb{Z}/N\mathbb{Z}} \overline{\chi(m)}\xi_N^{-m} \\ &= \sum_{m\in\mathbb{Z}/N\mathbb{Z}} G(m,\chi)\xi_N^{-m} = \sum_{n\in\mathbb{Z}/N\mathbb{Z}} \sum_{m\in\mathbb{Z}/N\mathbb{Z}} \chi(n)\xi_N^{mn}\xi_N^{-m} \\ &= \sum_{n\in\mathbb{Z}/N\mathbb{Z}} \chi(n) \sum_{m\in\mathbb{Z}/N\mathbb{Z}} \xi_N^{m(n-1)} = N\chi(1) = N, \end{aligned}$$

where in the second last step we have used that $\sum_{m\in\mathbb{Z}/N\mathbb{Z}} \xi_N^{m(n-1)} = 0$ if $n-1 \neq 0$.

2. Let χ be any non-trivial even Dirichlet character modulo a prime p. Define

$$\Theta(\chi,t) := \sum_{1\leq n} \chi(n)e^{-\pi t n^2} = \frac{1}{2}\sum_{n\in\mathbb{Z}} \chi(n)e^{-\pi t n^2}$$

for any real $t > 0$. For any real α and any real $t > 0$ define

$$\Theta_\alpha(t) := \sum_{n\in\mathbb{Z}} e^{-\pi t(n+\alpha)^2} \quad \text{and} \quad \Theta^\alpha(t) := \sum_{n\in\mathbb{Z}} e^{2\pi n\alpha}e^{-\pi t n^2}.$$

a) Show that

1. $\Theta(\chi,t) = \frac{1}{2}\sum_{1\leq a\leq p} \chi(a)\Theta_{\frac{a}{p}}(p^2 t)$,
2. $\frac{1}{2}\sum_{1\leq a\leq p} \chi(a)\Theta^{\frac{a}{p}}(t) = G(1,\chi)\Theta(\overline{\chi},t)$,
3. $\Theta(\chi,t) = \frac{G(1,\chi)}{\sqrt{p^2 t}}\Theta(\overline{\chi},\frac{1}{p^2 t})$.

Proof:

For Part 1 we write $\Theta_{\frac{a}{p}}(p^2 t) = \sum_{n\in\mathbb{Z}} e^{-\pi t(pn+a)^2}$ and then get

$$\begin{aligned} \sum_{1\leq a\leq p} \chi(a)\Theta_{\frac{a}{p}}(p^2 t) &= \sum_{1\leq a\leq p} \left(\sum_{n\in\mathbb{Z}} \chi(a)e^{-\pi t(pn+a)^2} \right) \\ &= \sum_{n\in\mathbb{Z}} \left(\sum_{1\leq a\leq p} \chi(a)e^{-\pi t(pn+a)^2} \right) \\ &= \sum_{n\in p\mathbb{Z}} \left(\sum_{1\leq a\leq p} \chi(n+a)e^{-\pi t(n+a)^2} \right) \\ &= \sum_{n\in\mathbb{Z}} \chi(n)e^{-\pi t n^2}, \end{aligned}$$

where we have used that $\chi(n+a) = \chi(a)$ for any $a \in \mathbb{Z}$ and any $n \in p\mathbb{Z}$.

Part 2 follows from

$$\sum_{1\le a\le p} \chi(a)\Theta^{\frac{a}{p}}(t) = \sum_{n\in\mathbb{Z}} \left(\sum_{1\le a\le p} \chi(a) e^{2\pi n \frac{a}{p}} \right) e^{-\pi t n^2} = \sum_{n\in\mathbb{Z}} G(n,\chi) e^{-\pi t n^2}$$
$$= \sum_{n\in\mathbb{Z}} \overline{\chi}(n) G(1,\chi) e^{-\pi t n^2} = 2G(1,\chi)\Theta(\overline{\chi}, t),$$

where we have used Part c) of Exercise 1.

For Part 3 we fix some $t > 0$ and consider $g(x) := e^{-\pi p^2 t x^2}$ for any $x \in \mathbb{R}$. The Fourier transform of g is $\hat{g}(y) = \frac{1}{\sqrt{p^2 t}} e^{\frac{-\pi y^2}{p^2 t}}$. Poisson summation then yields

$$\Theta_{\frac{a}{p}}(p^2 t) = \sum_{n\in\mathbb{Z}} e^{-\pi p^2 t (n+\frac{a}{p})^2} = \sum_{n\in\mathbb{Z}} g\left(n + \frac{a}{p}\right)$$
$$= \sum_{n\in\mathbb{Z}} \hat{g}(n) e^{2\pi i n \frac{a}{p}} = \frac{1}{\sqrt{p^2 t}} \sum_{n\in\mathbb{Z}} e^{2\pi i n \frac{a}{p}} e^{\frac{-\pi n^2}{p^2 t}}.$$

Using Part 1 and Exercise 1, we thus get

$$2\Theta(\chi, t) = \sum_{1\le a\le p} \chi(a)\Theta_{\frac{a}{p}}(p^2 t) = \frac{1}{\sqrt{p^2 t}} \sum_{n\in\mathbb{Z}} \left(\sum_{1\le a\le p} \chi(a) e^{2\pi i n \frac{a}{p}} \right) e^{\frac{-\pi n^2}{p^2 t}}$$
$$= \frac{1}{\sqrt{p^2 t}} \sum_{n\in\mathbb{Z}} G(n,\chi) e^{\frac{-\pi n^2}{p^2 t}} = \frac{G(1,\chi)}{\sqrt{p^2 t}} \sum_{n\in\mathbb{Z}} \overline{\chi}(n) e^{\frac{-\pi n^2}{p^2 t}}$$
$$= 2\frac{G(1,\chi)}{\sqrt{p^2 t}} \Theta(\overline{\chi}, \frac{1}{p^2 t}).$$

as desired.

b) Let $L(\chi, s) := \sum_{n\ge 1} \frac{\chi(n)}{n^s}$ be the L-series associated with χ. Show that for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}$ we have

$$\pi^{-s}\Gamma(s)L(\chi, 2s) = \int_0^\infty \Theta(\chi, t) t^s \frac{dt}{t}.$$

Proof:

Let us use that

$$\int_0^\infty e^{-ct} t^s \frac{dt}{t} = \frac{\Gamma(s)}{c^s}$$

for any positive real number c and any $s \in \mathbb{C}$. By Fubini's theorem we thus get

$$\int_0^\infty \Theta(\chi, t) t^s \frac{dt}{t} = \int_0^\infty \sum_{1\le n} \chi(n) e^{-\pi t n^2} t^s \frac{dt}{t} = \sum_{1\le n} \chi(n) \int_0^\infty e^{-\pi t n^2} t^s \frac{dt}{t}$$
$$= \sum_{1\le n} \frac{\chi(n)\Gamma(s)}{n^{2s}\pi^s} = \pi^{-s}\Gamma(s)L(\chi, 2s).$$

c) Show that $L(\chi, \cdot)$ has an analytic continuation to $\mathbb{C}$ which is entire and find a functional equation that relates $L(\chi, s)$ with $L(\overline{\chi}, 1-s)$.

Proof:

Since $\Theta(\chi, t)$ decays exponentially as $t \to \infty$, the integral $\int_0^\infty \Theta(\chi, t) t^s \frac{dt}{t}$ converges absolutely and locally uniformly in s and thus defines an analytic function on $\mathbb{C}$. Using the third part of Part a), we get that

$$\int_0^\infty \Theta(\chi, t) t^s \frac{dt}{t} = \frac{G(1,\chi)}{p} \int_0^\infty \Theta(\overline{\chi}, \frac{1}{p^2 t}) t^{s-\frac{1}{2}} \frac{dt}{t} = G(1,\chi) p^{-2s} \int_0^\infty \Theta(\overline{\chi}, \tilde{t}) \tilde{t}^{\frac{1}{2}-s} \frac{d\tilde{t}}{\tilde{t}},$$

where we have obtained the second equation through the change of variables $\tilde{t} := \frac{1}{p^2 t}$. By Part b) this implies

$$\pi^{-s}\Gamma(s) L(\chi, 2s) = G(1,\chi) p^{-2s} \pi^{s-\frac{1}{2}} \Gamma(\frac{1}{2} - s) L(\overline{\chi}, 1-2s).$$

Setting $\Lambda(\chi, s) := \left(\frac{p}{\pi}\right)^{\frac{s}{2}} \Gamma(\frac{s}{2}) L(\chi, s)$, this reads as the functional equation

$$\Lambda(\chi, s) = \frac{G(1,\chi)}{\sqrt{p}} \Lambda(\overline{\chi}, 1-s)$$

which in particular relates $L(\chi, s)$ to $L(\overline{\chi}, 1-s)$.

3. Let $\Gamma_\vartheta := \langle T^2, S \rangle = \langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle \subset SL_2(\mathbb{Z})$.

a) Show that

$$\begin{aligned} \Gamma_\vartheta &= \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mod 2 \right\} \\ &= \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, a+b+c+d \equiv 0 \mod 2 \right\} \\ &= \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, ab \equiv cd \equiv 0 \mod 2 \right\}. \end{aligned}$$

In particular, $\Gamma(2) \subset \Gamma_\vartheta \subset SL_2(\mathbb{Z})$. Hence Γ_ϑ is a congruence subgroup of level 2.

Proof: The second and the third equality can be checked straightforwardly. In order to see the first equality we set

$$\Gamma'_\vartheta := \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mod 2 \right\}.$$

It is immediately seen that $\Gamma_\vartheta \subset \Gamma'_\vartheta$. For the reverse inclusion we first find two suitable fundamental domains for Γ'_ϑ. We have $SL_2(\mathbb{Z}/2\mathbb{Z}) = \{[1], [S], [T], [ST], [TS], [STS]\}$ and it is immediately checked that $[T]$ and $[TS]$ form a set of representatives for

$$SL_2(\mathbb{Z}/2\mathbb{Z})/\{1, [S]\}.$$

As the reduction homomorphism

$$SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/2\mathbb{Z}) \to SL_2(\mathbb{Z}/2\mathbb{Z})/\{1, [S]\}$$

induces an isomorphism $SL_2(\mathbb{Z})/\Gamma'_\vartheta \cong SL_2(\mathbb{Z}/2\mathbb{Z})/\{1, [S]\}$, we therefore get that T and TS form a set of representatives for $SL_2(\mathbb{Z})/\Gamma'_\vartheta$. Thus $\mathcal{F}_\vartheta := \mathcal{F} \cup T\mathcal{F} \cup TS\mathcal{F}$ is a fundamental domain for Γ'_ϑ and the cusps of Γ'_ϑ are $[i\infty]$, $T[i\infty] = [i\infty]$ and $TS[i\infty] = [1]$. Then also

$$\tilde{\mathcal{F}}_\vartheta := \{z \in \mathbb{H} \mid |z| \geq 1 \text{ and } -1 \leq \mathrm{Re}(z) \leq 1\}$$

is a fundamental domain for Γ'_ϑ since $\tilde{\mathcal{F}}_\vartheta$ is obtained from $\mathcal{F}_\vartheta$ by shifting the piece $\mathcal{F}_\vartheta \cap \{z \in \mathbb{H} \mid \mathrm{Re}(z) \geq 1\}$ to $\tilde{\mathcal{F}}_\vartheta \cap \{z \in \mathbb{H} \mid \mathrm{Re}(z) \leq \frac{1}{2}\}$ via $(T^2)^{-1}$.

We claim that $\tilde{\mathcal{F}}_\vartheta$, and hence also $\mathcal{F}_\vartheta$, is a fundamental domain for Γ_ϑ too. Since $\Gamma_\vartheta \subset \Gamma'_\vartheta$, it is enough to show that for any $z \in \mathbb{H}$ there exists a $\gamma \in \Gamma_\vartheta$ such that $\gamma z \in \tilde{\mathcal{F}}_\vartheta$. We sketch a proof of this which is analogous to the proof given in the lecture of the fact that $\mathcal{F}$ is a fundamental domain for $SL_2(\mathbb{Z})$. Namely, by a lemma of the lecture there exists a z' in the Γ_ϑ-orbit of z such that $\mathrm{Im}(z')$ is maximal among the imaginary parts of elements in this orbit. Moreover, there exists an integer n such that $z'' := (T^2)^n z'$ satisfies $-1 \leq \mathrm{Re}(z'') \leq 1$. In particular, $\mathrm{Im}(z'') = \mathrm{Im}(z')$. We also have that $|z''| \geq 1$ since otherwise the imaginary part $\mathrm{Im}(Sz'') = \frac{\mathrm{Im}(z'')}{|z''|^2} = \frac{\mathrm{Im}(z')}{|z''|^2}$ of the element Sz'' in the Γ_ϑ-orbit of z would be strictly greater than that of z' and hence contradict the maximality of the latter. Thus $z'' \in \tilde{\mathcal{F}}_\vartheta$ which proves the claim.

Consider finally any $\gamma' \in \Gamma'_\vartheta$ and choose any z in the interior of $\mathcal{F}$. Since $\mathcal{F}_\vartheta$ is a fundamental domain for Γ_ϑ, there exists $\gamma \in \Gamma_\vartheta$ such that $\gamma\gamma' z \in \mathcal{F}_\vartheta$. Hence $\gamma\gamma' z \in \mathcal{F}$ or $T^{-1}\gamma\gamma' z \in \mathcal{F}$ or $(ST)^{-1}\gamma\gamma' z \in \mathcal{F}$. Since $\mathcal{F}$ is a fundamental domain for $SL_2(\mathbb{Z})$ and since z is in the interior of $\mathcal{F}$, we conclude that $\gamma\gamma' = \pm 1$ or $T^{-1}\gamma\gamma' = \pm 1$ or $(ST)^{-1}\gamma\gamma' = \pm 1$. We have $-1 = S^2 \in \Gamma_\vartheta$ but $T \notin \Gamma'_\vartheta$ and $ST \notin \Gamma'_\vartheta$ so that the only possible case is $\gamma\gamma' = \pm 1$. Thus $\gamma' = \pm\gamma^{-1} \in \Gamma_\vartheta$. This shows that $\Gamma'_\vartheta \subset \Gamma_\vartheta$.

b) Show that

$$SL_2(\mathbb{Z}) = \Gamma_\vartheta \,\dot{\cup}\, \Gamma_\vartheta T \,\dot{\cup}\, \Gamma_\vartheta TS$$

and draw a picture of a fundamental domain of Γ_ϑ. Show that Γ_ϑ has the two cusps $[i\infty]$ and $[1] = TS[i\infty]$.

Proof: These assertions were shown in Part a) already. A picture of the fundamental domain $\mathcal{F}_\vartheta := \mathcal{F} \cup T\mathcal{F} \cup TS\mathcal{F}$ for Γ_ϑ can be found between A.5.2 and A5.3 in [1]. There is an online version of [1] on Springer Link.

c) Let k be an integer. Recall that a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is in $M_k(\Gamma_\vartheta)$ if and only if

1. $f|_k T^2 = f$,
2. $f|_k S = f$,
3. the limit $f(i\infty) := \lim_{y\to\infty} f(\tau)$ exists,
4. the limit $f(1) := \lim_{y\to\infty}(f|_k TS)(\tau) = \lim_{y\to\infty} \tau^{-k} f\left(1 - \frac{1}{\tau}\right)$ exists.

Show that there are no non-constant modular forms of weight zero for Γ_ϑ, i.e. that $M_0(\Gamma_\vartheta) = \mathbb{C}$.

Proof: Consider any $f \in M_0(\Gamma_\vartheta)$ and set $g(\tau) := (f(\tau) - f(i\infty))(f(\tau) - f(1))$. Then $g \in M_0(\Gamma_\vartheta)$ and $g(i\infty) = g(1) = 0$. Since $[i, \infty]$ and $[1]$ are the cusps of $\mathcal{F}_\vartheta$, we see that g is bounded on $\mathcal{F}_\vartheta$. Since g is invariant under Γ_ϑ it is therefore bounded on all of $\mathbb{H}$. By the maximum modulus principle, g is thus constant. Since g is zero at the cusps, we have $g = 0$. Therefore f attains at most two values. As $f(\tau)$ is continous and $\mathbb{H}$ connected, f in fact attains only one value, i.e. it is constant.

4. For any natural numbers $r \geq 1$ and $n \geq 0$ consider the number

$$A_r(n) := |\{x = (x_1, ..., x_r) \in \mathbb{Z}^r \mid x_1^2 + ... + x_r^2 = n\}|$$

of possibilities to write n as a sum of r squares of integers. The goal of this exercice is to prove Jacobi's 8-squares formula

$$A_8(n) = 16 \sum_{d|n} (-1)^{n-d} d^3.$$

To this end consider the theta-series $\vartheta(\tau) := \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$ and $\Theta(\tau) := \vartheta(2\tau)$ so that

$$\Theta^r(\tau) = \sum_{n=0}^{\infty} A_r(n) e^{2\pi n \tau}.$$

a) Show that ϑ satisfies

1. $\vartheta(\tau + 2) = \vartheta(\tau)$,
2. $\vartheta(\frac{-1}{\tau}) = \sqrt{\frac{\tau}{i}}\vartheta(\tau)$,
3. $\lim_{y \to \infty} \vartheta(\tau) = 1$,
4. $\lim_{y \to \infty} \left(\sqrt{\frac{\tau}{i}}^{-1} \vartheta(1 - \frac{1}{\tau}) e^{-\pi i \frac{\tau}{4}} \right) = 2$.

In particular, $\vartheta(i\infty) = 1$ and $\vartheta(1) = 0$.

Proof: First recall Jacobi's Theta transformation formula

$$\sqrt{\frac{\tau}{i}} \sum_{n \in \mathbb{Z}} e^{\pi i (n+w)^2 \tau} = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 (\frac{-1}{\tau})} e^{2\pi i n w}$$

for any $w \in \mathbb{C}$ and any $\tau \in \mathbb{H}$, which was proved in the lecture. The case $w = 0$ is precisely the second of the four properties of ϑ which are to be checked. Properties 1 and 3 follow immediately from the definition. In order to see the fourth, we apply the transformation formula in the case $w = \frac{1}{2}$ to get

$$\begin{aligned} \sqrt{\frac{\tau}{i}} \sum_{n \in \mathbb{Z}} e^{\pi i (n+\frac{1}{2})^2 \tau} &= \sum_{n \in \mathbb{Z}} e^{\pi i n^2 (\frac{-1}{\tau})} e^{\pi i n} = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 (1 - \frac{1}{\tau})} \left(e^{-\pi i n^2} e^{\pi i n} \right) \\ &= \sum_{n \in \mathbb{Z}} e^{\pi i n^2 (1 - \frac{1}{\tau})} = \vartheta\left(1 - \frac{1}{\tau}\right). \end{aligned}$$

This yields

$$\left(\sqrt{\frac{\tau}{i}}\right)^{-1}\vartheta(1-\frac{1}{\tau})e^{-\pi i\frac{\tau}{4}} = q^{-\frac{1}{8}}\left(\sum_{n\in\mathbb{Z}} q^{\frac{\left(n+\frac{1}{2}\right)^2}{2}}\right) = q^{-\frac{1}{8}}\left(2q^{\frac{1}{8}} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0,-1}} q^{\frac{\left(n+\frac{1}{2}\right)^2}{2}}\right)$$
$$= 2 + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0,-1}} q^{\frac{\left(n+\frac{1}{2}\right)^2}{2}},$$

where $q = e^{2\pi i\tau}$. From this, the fourth property follows immediately.

b) Show that ϑ does not vannish on $\mathbb{H}\cup\{i\infty\}$.

Proof: By the third property of Part a) we have $\vartheta([i\infty]) = 1$. Let us check that ϑ does not vanish on $\mathbb{H}$. By the transformation property of ϑ it is enough to show that ϑ does not vanish on its fundamental domain $\mathcal{F}_\vartheta$ defined in the solutions to Exercise 3. This in turn is equivalent to showing that $\vartheta(\tau), \vartheta(\tau+1)$ and $\vartheta(1-\frac{1}{\tau})$ are non-zero for any τ in the fundamental domain $\mathcal{F}$ for $SL_2(\mathbb{Z})$. We thus consider any $\tau\in\mathcal{F}$ and use that $\mathrm{Im}(\tau)\geq\frac{\sqrt{3}}{2}$ to get

$$|\vartheta(\tau)-1| \leq 2\sum_{n=1}^{\infty} e^{-\pi n^2 y} \leq 2\sum_{n=1}^{\infty} e^{-\frac{\pi}{2}\sqrt{3}n^2} \leq 2\sum_{n=1}^{\infty} e^{-\frac{\pi}{2}\sqrt{3}n} = \frac{2e^{-\frac{\pi}{2}\sqrt{3}}}{1-e^{-\frac{\pi}{2}\sqrt{3}}} < 0.2$$

In particular we have $|\vartheta(\tau)|\geq 0.8 > 0$ for any $\tau\in\mathcal{F}$. Similarly, $|\vartheta(\tau+1)-1| < 0.2$ for any $\tau\in\mathcal{F}$. Let us finally consider $\vartheta\left(1-\frac{1}{\tau}\right)$ for any $\tau\in\mathcal{F}$. As shown in Part a) we have

$$\vartheta\left(1-\frac{1}{\tau}\right) = e^{\frac{\pi i\tau}{4}}\sqrt{\frac{\tau}{i}}\sum_{n\in\mathbb{Z}} e^{\pi i(n^2+n)\tau}.$$

As $e^{\frac{\pi i\tau}{4}}\sqrt{\frac{\tau}{i}}$ does not vanish on $\mathbb{H}$, we are reduced to showing that $\sum_{n\in\mathbb{Z}} e^{\pi i(n^2+n)\tau}\neq 0$ for any $\tau\in\mathcal{F}$. But this follows from

$$\left|2-\sum_{n\in\mathbb{Z}} e^{\pi i(n^2+n)\tau}\right| \leq \sum_{n=1}^{\infty} e^{-\frac{\pi}{2}\sqrt{3}n(n+1)} + \sum_{n=2}^{\infty} e^{-\frac{\pi}{2}\sqrt{3}n(n-1)} = 2\sum_{n=1}^{\infty} e^{-\frac{\pi}{2}\sqrt{3}n(n+1)}$$
$$\leq 2\sum_{n=1}^{\infty} e^{-\frac{\pi}{2}\sqrt{3}n} \leq \frac{2e^{-\frac{\pi}{2}\sqrt{3}}}{1-e^{-\frac{\pi}{2}\sqrt{3}}} < 0.2$$

c) Suppose that $r\equiv 0$ modulo 4. Let $f\in M_{\frac{r}{2}}(\Gamma_\vartheta)$ such that

$$\lim_{y\to\infty}\left(\left(\sqrt{\frac{\tau}{i}}\right)^{-r} f(1-\frac{1}{\tau})e^{-\pi i\frac{\tau r}{4}}\right)$$

exists. Note that this condition is stronger than the condition at the cusp [1] in the Definition of $M_{\frac{r}{2}}(\Gamma_\vartheta)$. Show that then $f = c\vartheta^r$ for some constant $c\in\mathbb{C}$.

Hint: Use the last part of Exercise 3.

Proof: We claim that $\frac{f}{\vartheta^r} \in M_0(\Gamma_\vartheta)$. It must then be constant by Part c) of Exercise 3. By definition of f and by Part a), $\frac{f}{\vartheta^r}$ is modular of weight 0 with respect to Γ_ϑ. From Part b) we get that $\frac{f}{\vartheta^r}$ is holomorphic on $\mathbb{H}$ and at $[i\infty]$. It remains to be shown that $\frac{f}{\vartheta^r}$ is holomorphic at $[1]$. But this follows immediately from the fourth property in Part a), because that property implies that $\lim_{y\to\infty}\left(\left(\sqrt{\frac{\tau}{i}}\right)^{-r}\vartheta(1-\frac{1}{\tau})e^{-\pi i\frac{\tau r}{4}}\right)$ exists. The limit is in fact 2^r

d) Consider the Eisenstein series

$$G_k(\tau) := \sum_{(0,0)\neq(c,d)\in\mathbb{Z}^2}\frac{1}{(c\tau+d)^k}$$

and consider any $a, b \in \mathbb{C}$. Show that $f(\tau) := aG_k(\tau) + bG_k(\frac{\tau+1}{4}) \in M_k(\Gamma_\vartheta)$. Moreover, show that if $k = 4$ and $a = -16b$, then $f = c\vartheta^8$ for some $c \in \mathbb{C}$.

Proof: We have already seen that $G_k \in M_k(SL_2(\mathbb{Z})) \subset M_k(\Gamma_\vartheta)$ with $G_k(i\infty) = 2\zeta(k)$. In order to see the first claim we are thus reduced to showing that $G_k\left(\frac{\tau+1}{2}\right) \in M_k(\Gamma_\vartheta)$. It is immediately seen, that $G_k\left(\frac{\tau+1}{2}\right)$ satisfies the first and the third property in Part c) in Exercise 3 with $\lim\limits_{y\to\infty} G_k\left(\frac{\tau+1}{2}\right) = \lim\limits_{y\to\infty} G_k(\tau) = 2\zeta(k)$. In order to see the fourth property, we use that

$$\tau^{-k}G_k\left(\frac{1-\frac{1}{\tau}+1}{2}\right) = \tau^{-k}G_k(-\frac{1}{2\tau}) = 2^kG_k(2\tau) \tag{2}$$

and get $\lim\limits_{y\to\infty} G_k\left(\frac{1-\frac{1}{\tau}+1}{2}\right) = \lim\limits_{y\to\infty} 2^kG_k(2\tau) = 2^{k+1}\zeta(k)$. In order to see the second property, we compute

$$\begin{aligned} G_k\left(\frac{\frac{-1}{\tau}+1}{2}\right) &= G_k\left(\frac{1}{2}-\frac{1}{2\tau}\right) = \sum_{m,n\in\mathbb{Z}}{}' \left(m\left(\frac{1}{2}-\frac{1}{2\tau}\right)+d\right)^{-k} \\ &= \sum_{\substack{m',n'\in\mathbb{Z}\\ m'+n' \text{ even}}}{}' (m'\tau+n')^{-k}, \end{aligned}$$

where in the last step we have set $m' := m+2n$ and $n' := -m$, which induces a bijection from $\{0 \neq (m,n) \in \mathbb{Z}^2\}$ to $\{0 \neq (m',n') \in \mathbb{Z}^2 \mid m'+n' \text{ even}\}$. Similarly we get

$$G_k\left(\frac{\tau+1}{2}\right) = \sum_{m,n\in\mathbb{Z}}{}' (m\left(\frac{\tau+1}{2}\right)+n)^{-k} = \sum_{\substack{m',n'\in\mathbb{Z}\\ m'-n' \text{ even}}}{}' (m'\tau+n')^{-k},$$

where in the last step we have set $m' = m$ and $n' = m+2n$, which induces a bijection from $\{0 \neq (m,n) \in \mathbb{Z}^2\}$ to $\{0 \neq (m',n') \in \mathbb{Z}^2 \mid m'-n' \text{ even}\}$. This yields the needed equality

$$G_k\left(\frac{\frac{-1}{\tau}+1}{2}\right) = \tau^kG_k\left(\frac{\tau+1}{2}\right).$$

Let us finally show that for any $b \in C$ the modular form $16bG_4(\tau) - bG_4(\frac{\tau+1}{2})$ is a scalar multiple of ϑ^8. Without loss of generality we may restrict ourselves to the case where $b = 1$. Set $f(\tau) := 16G_4(\tau) - G_4(\frac{\tau+1}{2})$. By Part c) it is enough to show that

$$\lim_{y \to \infty} \left(\tau^{-4} f(1 - \frac{1}{\tau}) q(\tau)^{-1} \right)$$

exists, where $q(\tau) := e^{2\pi i \tau}$. By (2), we have

$$\begin{aligned}
\tau^{-4} f(1 - \frac{1}{\tau}) q(\tau)^{-1} &= \left(16G_4(\tau) - 2^4 G_4(2\tau)\right) q(\tau)^{-1} \\
&= 16 \cdot 2 \cdot \zeta(4) \cdot 240 \sum_{n=1}^{\infty} \left(\sigma_3(n) q^{n-1} - \sigma_3(n) q^{2n-1}\right).
\end{aligned}$$

The limit of $\tau^{-4} f(1 - \frac{1}{\tau}) q(\tau)^{-1}$ as $y \to \infty$ thus exists. More precisely, this limit is $16 \cdot 2 \cdot \zeta(4) \cdot 240 = \frac{16^2 \pi^4}{3}$. On the other hand, the limit of $\tau^{-4} \vartheta^8 (1 - \frac{1}{\tau}) q(\tau)^{-1}$ as $y \to \infty$ is 2^8 by Part a). Therefore the constant $\frac{f}{\vartheta^8}$ is $\frac{16^2 \pi^4}{3} \frac{1}{2^8} = \frac{\pi^4}{3}$.

e) Show that

$$\vartheta^8 = \frac{3}{\pi^4} \left(16 G_4(\tau) - G_4(\frac{\tau+1}{2})\right)$$

and conclude that

$$A_8(n) = 16 \sum_{d|n} (-1)^{n-d} d^3.$$

Proof: In Part d) we have already shown the first equality. Using this equality and the Fourier expansion of G_4 we conclude that

$$\begin{aligned}
1 + \sum_{n=1}^{\infty} A_8(n) e^{\pi i n \tau} &= \vartheta^8(\tau) = \frac{3}{\pi^4} f(\tau) \\
&= 1 + 16^2 \sum_{n=1}^{\infty} \sigma_3(n) e^{2\pi i n \tau} - 16 \sum_{n=1}^{\infty} \sigma_3(n) (-1)^n e^{\pi i n \tau}.
\end{aligned}$$

By comparison of the Fourier coefficients we thus see for any odd positive integer that

$$A_8(n) = 16 \sigma_3(n) = 16 \sum_{d|n} (-1)^{n-d} d^3$$

and get for any even positive integer as well that

$$\begin{aligned}
A_8(n) = 16^2 \sigma_3\left(\frac{n}{2}\right) - 16\sigma_3(n) &= 16^2 \sum_{2d|n} d^3 - 16 \sum_{d|n} d^3 \\
= 16 \left(\sum_{\substack{d|n \\ d \text{ even}}} 2d^3 - \sum_{d|n} d^3 \right) &= 16 \left(\sum_{\substack{d|n \\ d \text{ even}}} d^3 - \sum_{\substack{d|n \\ d \text{ odd}}} d^3 \right) \\
= 16 \sum_{d|n} (-1)^d d^3 = 16 \sum_{d|n} (-1)^{n-d} d^3.
\end{aligned}$$

Literatur

[1] Freitag, E.; Busam, R. *Funktionentheorie 1*, Springer-Verlag Berlin Heidelberg 1993, pp 369-372.