Solution sheet 6

1. Set $\Gamma := SL_2(\mathbb{Z})$ and let $\alpha \in GL_2^+(\mathbb{Q})$.

a) Show that the subgroup $\Gamma_1 := \alpha^{-1}\Gamma\alpha \cap \Gamma$ is a congruence subgroup.

Proof: It is enough to show that $\alpha^{-1}\Gamma\alpha$ contains a principal subgroup. Choose a large enough positive integer N such that $N\alpha, N\alpha^{-1} \in \mathrm{M}_2(\mathbb{Z})$. We claim that $\Gamma(N^2) \subset \alpha^{-1}\Gamma\alpha$. Any $\gamma \in \Gamma(N^2)$ is of the form $\mathrm{id} + N^2 g$ for some $g \in M_2(\mathbb{Z})$ and hence

$$\alpha\gamma\alpha^{-1} = \mathrm{id} + (N\alpha)g(N\alpha^{-1})$$

has integer entries and determinant 1 so that $\alpha\gamma\alpha^{-1} \in \Gamma$. This shows that $\alpha\Gamma(N^2)\alpha^{-1} \subset \Gamma$ and, equivalently, $\Gamma(N^2) \subset \alpha^{-1}\Gamma\alpha$.

b) Show that for any $f, g \in S_k(\Gamma)$ we have that $f|_k\alpha, g|_k\alpha \in S_k(\Gamma_1)$ and that

$$\langle f, g\rangle = \langle f|_k\alpha, g|_k\alpha\rangle,$$

where the latter inner product is with respect to Γ_1.

Proof:

Consider any $\tau \in \Gamma_1$. Then there exists $\gamma \in \Gamma$ such that $\tau = \alpha^{-1}\gamma\alpha$ and hence

$$(f|_k\alpha)|_k\tau = f|_k(\alpha\tau) = f|_k(\gamma\alpha) = (f|_k\gamma)|_k\alpha = f|_k\alpha$$

since $f|_k\gamma = f$ by assumption. In order to see the second assertion let $\Gamma_2 := \Gamma \cap \alpha\Gamma\alpha^{-1}$ so that $\alpha^{-1}\Gamma_2\alpha = \Gamma_1$. Let $\mathcal{F}_2$ be a fundamental domain for Γ_2. Then $\alpha^{-1}\mathcal{F}_2$ is a fundamental domain for Γ_1. Setting $\overline{\Gamma_1} := \{\pm \mathrm{id}\}\Gamma_1 \subset \Gamma$ and denoting by $\mathcal{F}$ the standard fundamental domain for Γ we thus get

$$\begin{aligned}
\langle f|_k\alpha, g|_k\alpha\rangle &:= \frac{1}{|SL_2(\mathbb{Z}) : \overline{\Gamma_1}|} \int\int_{\alpha^{-1}\mathcal{F}_2} (f|_k\alpha)(z)\overline{(g|_k\alpha))(z)}y^k \frac{dxdy}{y^2} \\
&= \frac{1}{|SL_2(\mathbb{Z}) : \overline{\Gamma_1}|} \int\int_{\mathcal{F}_2} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2} \\
&= \frac{1}{|SL_2(\mathbb{Z}) : \overline{\Gamma_2}|} \int\int_{\mathcal{F}_2} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2} \\
&= \int\int_{\mathcal{F}} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2} \\
&= \langle f, g\rangle
\end{aligned}$$

by the invariance of the inner product and since

$$|SL_2(\mathbb{Z}) : \overline{\Gamma_1}| = |SL_2(\mathbb{Z}) : \overline{\alpha^{-1}\Gamma_2\alpha}| = |SL_2(\mathbb{Z}) : \overline{\Gamma_2}|.$$

2. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, fix $\alpha \in GL_2^+(\mathbb{Q})$ and set $\Gamma_1 := \alpha^{-1}\Gamma\alpha \cap \Gamma$.

a) Consider the double coset $\Gamma\alpha\Gamma$ in $\mathrm{GL}_2^+(\mathbb{Q})$. There is a natural group action by Γ on $\Gamma\alpha\Gamma$ by left matrix multiplication. Show that the assignment

$$\Gamma \to \Gamma\alpha\Gamma, \quad \gamma \mapsto \alpha\gamma$$

yields a one-to-one correspondence between $\Gamma\backslash\Gamma\alpha\Gamma$ and $\Gamma_1\backslash\Gamma$. By Part 1**a)**, it follows then that the orbit space $\Gamma\backslash\Gamma\alpha\Gamma$ is finite.

Proof:

The assignment $\gamma \mapsto \alpha\gamma$ induces a surjective map from Γ to $\Gamma\backslash\Gamma\alpha\Gamma$. Consider moreover any $\gamma_1, \gamma_2 \in \Gamma$ such that $\Gamma\alpha\gamma_1 = \Gamma\alpha\gamma_2$. Then we have

$$\gamma_1\gamma_2^{-1} \in \alpha^{-1}\Gamma\alpha \cap \Gamma = \Gamma_1$$

as needed.

Let the slash operator for $\mathrm{GL}_2^+(\mathbb{Q})$ be defined as

$$f|_k[\gamma](z) = (\det\gamma)^{k-1} j(\gamma, z)^{-k} f(\gamma z)$$

for any $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$. Let $f \in \mathcal{M}_k(\Gamma)$. Consider the double coset slash operator given by

$$f|_k[\Gamma\alpha\Gamma] := \sum_{[\gamma]\in\Gamma\backslash\Gamma\alpha\Gamma} f|_k[\gamma].$$

b) Check that the above double coset operator is well-defined. Then show that the assignment $f \mapsto f|_k[\Gamma\alpha\Gamma]$ defines an operator

$$\mathcal{M}_k(\Gamma) \to \mathcal{M}_k(\Gamma), \quad \text{resp.} \quad \mathcal{S}_k(\Gamma) \to \mathcal{S}_k(\Gamma).$$

Proof:

Consider any complete sets of representatives R and R' for $\Gamma\backslash\Gamma\alpha\Gamma$ and any $f \in M_k(\Gamma)$. In order to see that the operator is well defined we need to show that

$$\sum_{\gamma\in R} f|_k\gamma = \sum_{\gamma'\in R'} f|_k\gamma'.$$

For any $\gamma \in R$ there exist unique $\gamma' \in R'$ and $\sigma \in \Gamma$ such that $\gamma = \sigma\gamma'$. Since $f \in M_k(\Gamma)$, we have

$$f|_k\gamma = f|_k(\sigma\gamma') = (f|_k\sigma)|_k\gamma' = f|_k\gamma'$$

from which the equality of the sums follows.

Moreover, for any $g \in \Gamma$ the set Rg is again a complete set of representatives for $\Gamma\backslash\Gamma\alpha\Gamma$. From this and the previously shown statement we conclude that

$$(f|_k[\Gamma\alpha\Gamma])\,|_k g = f|_k[\Gamma\alpha\Gamma]$$

for any $g \in \Gamma$. Finally, it is immediately checked that any summand of $f|_k[\Gamma\alpha\Gamma]$ is holomorphic at all cusps and vanishes at all cusps if f does. From this the exercise follows.

3. We set $\Gamma := SL_2(\mathbb{Z})$ and let k be any integer. Let p be a prime number and consider

$$\alpha_p := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \in SL_2(\mathbb{Z}).$$

The p'th *Hecke operator* of weight k is the operator $T_p : M_k(\Gamma) \to M_k(\Gamma)$ that sends $f \in M_k(\Gamma)$ to $T_p(f) := p^{\frac{k}{2}-1} f|_k[\Gamma\alpha_p\Gamma]$.

a) Show that

$$\Gamma\alpha_p\Gamma = \{\alpha \in GL_2(\mathbb{Z}) \mid \det\alpha = p\} = \bigcup_{j=0}^{p-1} \Gamma \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \dot{\cup}\, \Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

and conclude that

$$T_p(f)(\tau) = \frac{1}{p} \sum_{j=0}^{p-1} f\left(\frac{\tau+j}{p}\right) + p^{k-1} f(p\tau)$$

for any $f \in M_k(\Gamma)$. This shows that the definition of the Hecke operator T_p given here agrees with the one given in class.

Proof:

By a Lemma shown in the lecture, a complete set of representatives for the action of $SL_2(\mathbb{Z})$ on $\{\alpha \in M_2(\mathbb{Z}) \mid \det\alpha = p\}$ is given by

$$R := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad = p,\ 0 \le b < d \right\} = \left\{ \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \,\middle|\, 0 \le j < p \right\} \cup \left\{ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Moreover, we have

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = -S \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} S \in \Gamma\alpha_p\Gamma$$

and

$$\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} T^j \in \Gamma\alpha_p\Gamma.$$

From this the first equality follows. Finally,

$$\begin{aligned}
(T_p f)(\tau) &= p^{\frac{k}{2}-1} \sum_{\gamma \in R} f|_k \gamma(\tau) \\
&= p^{\frac{k}{2}-1} \sum_{j=1}^{p-1} f|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} (\tau) + p^{\frac{k}{2}-1} f|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} (\tau) \\
&= p^{\frac{k}{2}-1} \sum_{j=1}^{p-1} p^{\frac{k}{2}} (0\tau + p)^{-k} f\left(\frac{\tau+j}{p}\right) + p^{\frac{k}{2}-1} p^{\frac{k}{2}} (0\tau+1)^{-k} f(p\tau) \\
&= \frac{1}{p} \sum_{j=1}^{p-1} f\left(\frac{\tau+j}{p}\right) + p^{k-1} f(p\tau).
\end{aligned}$$

b) For any any positive integer m and any $f = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma)$ consider

$$V_m(f) := \sum_{n=0}^{\infty} a_n q^{mn}$$

and

$$U_m(f) := \sum_{n=0}^{\infty} a_m n q^n.$$

Show that

$$T_p(f) = U_p(f) + p^{k-1} V_p(f) = \sum_{n=0}^{\infty} b_n q^n,$$

where $b_n := a_{pn} + p^{k-1} a_{\frac{n}{p}}$ and $a_{\frac{n}{p}} := 0$ if p does not divide n.

Proof:

We have

$$f(m\tau) = \sum_{n=0}^{\infty} a_n q^{mn} = V_m f(\tau)$$

and

$$\frac{1}{m} \sum_{j=0}^{m-1} f\left(\frac{\tau + j}{m}\right) = \frac{1}{m} \sum_{j=0}^{m-1} \sum_{n=0}^{\infty} a_n q^{\frac{n}{m}} e^{2\pi i j \frac{n}{m}}$$
$$= \sum_{n=0}^{\infty} a_n q^{\frac{n}{m}} \left(\frac{1}{m} \sum_{j=0}^{m-1} e^{2\pi i j \frac{n}{m}} \right) = \sum_{m|n} a_n q^{\frac{n}{m}} = U_m f(\tau),$$

where we have used that

$$\frac{1}{m} \sum_{j=0}^{m-1} e^{2\pi i j \frac{n}{m}} = \begin{cases} 1 & \text{if } m|n \\ 0 & \text{otherwise.} \end{cases}$$

By Part a) we thus get

$$(T_p f)(\tau) = \frac{1}{p} \sum_{j=0}^{p-1} f\left(\frac{\tau + j}{p}\right) + p^{k-1} f(p\tau) = U_p f(\tau) + p^{k-1} V_p f(\tau)$$
$$= \sum_{n=0}^{\infty} a_{pn} q^n + p^{k-1} \sum_{\substack{n \geq 0 \\ p|n}} a_{\frac{n}{p}} q^n,$$

from which the Excercise follows.

c) Show that

$$1 - T_p X + p^{k-1} X^2 = (1 - U_p X)(1 - p^{k-1} V_p X)$$

where both sides are regarded as polynomials in the variable X with coefficients in the Hecke algebra $\mathscr{H}$ which operates on the $\mathbb{C}$-subspace of $\mathbb{C}[[q]]$ formed by the q-expansions of elements $f \in M_k$.

Prove that the following formal identities hold:

$$\sum_{n=1}^{\infty} T_n n^{-s} = \left(\sum_{n=1}^{\infty} n^{k-1} V_n n^{-s}\right)\left(\sum_{n=1}^{\infty} U_n n^{-s}\right) \qquad T_n = \sum_{d|n} d^{k-1} V_d U_{\frac{n}{d}}$$

Proof:

It is immediately checked that $U_p V_p = \mathrm{id}$. Using Part b) we thus get

$$\begin{aligned} 1 - T_p X + p^{k-1} X^2 &= 1 - \left(U_p + p^{k-1} V_p\right) X + p^{k-1} X^2 \\ &= 1 - U_p X - p^{k-1} V_p X + p^{k-1} U_p V_p X^2 \\ &= (1 - U_p X)(1 - p^{k-1} V_p X). \end{aligned}$$

Since the association $n \mapsto T_n$ is multiplicative, we may use the first Exercise of Sheet 5 to get that formally

$$\sum_{n=1}^{\infty} T_n n^{-s} = \prod_{p\in\mathbb{P}} \frac{T_{p^n}}{p^{sn}}_{n\geq 1} = \prod_{p\in\mathbb{P}} \left(1 - T_p p^{-s} + p^{k-1-2s}\right)^{-1},$$

where the second equality is checked via a straightforward computation which uses that $T_{p^r} = T_p T_{p^{r-1}} - p^{k-1} T_{p^{r-2}}$ for any integer $r \geq 2$. We moreover have that

$$1 - T_p p^{-s} + p^{k-1-2s} = (1 - p^{k-1} V_p p^{-s})^{-1}(1 - U_p p^{-s})$$

by Part b). As is immediately checked, the associations $n \mapsto U_n$ and $n \mapsto n^{k-1} V_n$ are completely multiplicative. Exercise 1 of Sheet 5 thus implies that

$$\prod_{p\in\mathbb{P}} (1 - p^{k-1} V_p p^{-s})^{-1} \prod_{p\in\mathbb{P}} (1 - U_p p^{-s})^{-1} = \left(\sum_{n=1}^{\infty} n^{k-1} V_n n^{-s}\right)\left(\sum_{n=1}^{\infty} U_n n^{-s}\right).$$

We therefore get the desired formal identity

$$\sum_{n=1}^{\infty} T_n n^{-s} = \left(\sum_{n=1}^{\infty} n^{k-1} V_n n^{-s}\right)\left(\sum_{n=1}^{\infty} U_n n^{-s}\right).$$

Finally, we write the right hand side of the above equation in terms of one sum and get

$$\begin{aligned} \sum_{n=1}^{\infty} T_n n^{-s} &= \left(\sum_{n=1}^{\infty} n^{k-1} V_n n^{-s}\right)\left(\sum_{n=1}^{\infty} U_n n^{-s}\right) = \sum_{n,m\in\mathbb{Z}_{>0}} n^{k-1} V_n U_m (nm)^{-s} \\ &= \sum_{n=1}^{\infty} \sum_{m|n} \left(\frac{n}{m}\right)^{k-1} V_{\frac{n}{m}} U_m n^{-s} = \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{k-1} V_d U_{\frac{n}{d}}\right) n^{-s}, \end{aligned}$$

where in the second last equality we have replaced nm by n and the condition $m > 0$ by $m|n$. This yields the second formal identity.

4. **a)** Let $k \geq 4$ be an even integer and let d be the $\mathbb{C}$-dimension of $S_k(SL_2(\mathbb{Z}))$. Choose any non-negative integers a, b such that $12 \neq 4a + 6b \leq 14$ and $4a + 6b = k \mod 12$. For each $1 \leq j \leq d$, define

$$f_j := \Delta^j E_6^{2(d-j)+b} E_4^a = \sum_{1 \leq n < \infty} a_n^{(j)} q^n,$$

where Δ is the normalized discriminant function and E_4 and E_6 are the normalized Eisenstein series of weight 4 respectively 6. Verify that $a_n^{(j)} = 0$ for $n < j$ and $a_j^{(j)} = 1$. Conclude that the f_j form a basis for $S_k(SL_2(\mathbb{Z}))$. This basis is called the *Miller basis*. Show moreover, that a modular form in $S_k(SL_2(\mathbb{Z})$ has integral Fourier coefficients if and only if it is a $\mathbb{Z}$-linear combination of the Miller basis.

Proof:

Let $\Delta = \sum_{0 \leq n < \infty} a_n q^n$ and $E_4 = \sum_{0 \leq n < \infty} b_n q^n$ and $E_6 = \sum_{0 \leq n < \infty} c_n q^n$ denote the Fourier expansions of the respective forms. We know that $a_0 = 0$ and $a_1 = b_0 = c_0 = 1$. From this we see that $a_n^{(j)} = 0$ for any $n < j$ and $a_j^{(j)} = 1$. This implies that the f_j's are $\mathbb{C}$-linearly independent. Moreover, the weight of any of the f_j is

$$12j + 12(d - j) + 4a + 6b = 12d + 4a + 6b$$

which we claim is k so that indeed $f_j \in S_k(\Gamma)$ for any $1 \leq j \leq d$. By a theorem shown in the lecture, we have $d = \lfloor \frac{k}{12} \rfloor - 1$ if $k = 2 \mod 12$ and $d = \lfloor \frac{k}{12} \rfloor$ otherwise. Using that

$$\lfloor \frac{k}{12} \rfloor = \frac{k - 4a - 6b}{12} + \lfloor \frac{4a + 6b}{12} \rfloor$$

and that $12 \neq 4a + 6b \leq 14$, the claim thus follows.

Let us turn to the second assertion of this exercise. By construction, any of the f_j has integral Fourier coefficients. In order to see the converse, we use Gaussian elimination to find a basis $g_1, \ldots, g_d$ of $S_k(\Gamma)$ such that for any $1 \leq i, j \leq d$ the i'th Fourier coefficient of g_j is $\delta_{i,j}$. Then the transformation matrix from the basis $f_1, \ldots, f_d$ to the basis $g_1, \ldots, g_d$ has integral coefficients and determinant 1. Consequently, any of the g_j is a $\mathbb{Z}$-linear combination of the f_j. Consider now any $f \in S_k(\Gamma)$ with integral coefficients. By construction of the g_j, we have that f is a $\mathbb{Z}$-linear combination of the g_j. Hence f is also a $\mathbb{Z}$-linear combination of the f_j as desired.

b) Let f be a normalized Hecke eigenform for $SL_2(\mathbb{Z})$. Show that the Fourier coefficients of f are algebraic integers.

Proof:

Since f is a normalized Hecke eigenfunction, its Fourier coefficients are eigenvalues of Hecke operators. We thus show that the eigenvalues of the n'th Hecke operator T_n are algebraic integers for any integer $n \geq 1$. Let k denote the weight of f and let $f_1, \ldots, f_d$ denote the Miller basis for $S_k(\Gamma)$. By construction, any of the f_j has integral coefficients. Thus also $T_n(f_j)$ has integral coefficients by the explicite formula shown in the class for the coefficients of $T_n(f_j)$ in terms of the coefficients of f_j. By Part a), $T_n(f_j)$ is thus a $\mathbb{Z}$-linear combination of the $f_1, \ldots, f_d$. Hence the matrix representing T_n with respect to the Miller basis has entries in $\mathbb{Z}$. Consequently, the characteristic polynomial of T_n is monic with coefficients in $\mathbb{Z}$. The eigenvalues of T_n are therefore indeed algebraic integers.