Exercise Sheet 1

1. a) Show that for all integers $n \geq 1$,

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Furthermore, deduce that

$$\sum_{d^2|n} \mu(d) = \begin{cases} 1 & \text{if } n \text{ is squarefree} \\ 0 & \text{otherwise.} \end{cases}$$

b) Prove that for $n \geq 1$,

$$\sum_{d|n} \varphi(d) = n,$$

where φ denotes the Euler totient function.

c) Verify the formula

$$\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d},$$

where $n \geq 1$.

2. The Dirichlet product $f * g$ of two arithmetic functions f and g is given by

$$(f * g)(n) = \sum_{d|n} f(d) g\Big(\frac{n}{d}\Big).$$

a) Check that for all arithmetic functions f, g, h the following two identities

$$f * g = g * f \qquad\qquad (f * g) * k = f * (g * k)$$

hold, i.e., check that the Dirichlet multiplication is commutative and associative.

b) Verify the Möbius inversion formula: for any two arithmetic functions f and g, we have

$$f(n) = \sum_{d|n} g(d) \quad \Leftrightarrow \quad g(n) = \sum_{d|n} f(d) \mu\Big(\frac{n}{d}\Big).$$

Bitte wenden!

3. The goal of this exercise is to recall the inclusion-exclusion principle.

a) Let $A_1, \dots, A_n$ be finite sets. Show that

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\substack{I\subset\{1,\dots,n\}\\ I\neq\emptyset}} (-1)^{|I|-1}\left|\bigcap_{i\in I} A_i\right|.$$

b) Let $A_1, \dots, A_n$ be measurable sets. Prove that

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\substack{I\subset\{1,\dots,n\}\\ I\neq\emptyset}} (-1)^{|I|-1}\mathbb{P}\left(\bigcap_{i\in I} A_i\right)$$

4. Recall the functional equation for the Riemann zeta function: for all $s \in \mathbb{C}$,

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s),$$

where

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx.$$

a) Show that for every positive integer k,

$$\zeta(2k) = (-1)^{k+1}\frac{(2\pi)^{2k}B_{2k}}{2(2k)!},$$

where B_n are the Bernoulli numbers.

b) Conclude that

$$\sum_{d\geq 1}\frac{1}{d^2} = \frac{\pi^2}{6}.$$

5. Show that for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$,

$$\sum_{d\geq 1}\frac{\mu(d)}{d^s} = \frac{1}{\zeta(s)}.$$

Submission: Monday, 28th September 2015 during the exercise class.