Exercise Sheet 10

1. Let

$$\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

The goal of this exercise is to prove the functional equation

$$\Lambda(s) = \Lambda(1-s). \tag{1}$$

a) Let

$$\theta(x) = \sum_{n\in\mathbb{Z}} e^{\pi n^2 x}.$$

Show that

$$\theta(x) - 1 \ll e^{-\pi x}$$

for all $x \geq 1$.

b) Prove that

$$\theta(x) = \frac{1}{\sqrt{x}}\theta\big(x^{-1}\big).$$

Hint: Use the Poisson summation formula.

c) Show that

$$\Lambda(s) = \frac{1}{2}\int_0^{+\infty} \big(\theta(y)-1\big)y^{\frac{s}{2}-1}dy$$

for $\mathrm{Re}(s) > 2$.

d) Show that

$$\frac{1}{2}\int_0^{+\infty} \big(\theta(y)-1\big)y^{\frac{s}{2}-1}dy = \frac{-1}{s(1-s)} + \int_1^{\infty} \big(\theta(x)-1\big)x^{\frac{s}{2}}\frac{dx}{x} + \int_1^{\infty} \big(\theta(x)-1\big)x^{\frac{1-s}{2}}\frac{dx}{x}.$$

Hint: Use part b) and standard integral transformations.

e) Conclude that $\Lambda(s)$ is a meromorphic function on $\mathbb{C}$ with simple poles at 0 and 1 satisfying the functional equation (1).
Hint: Use part a).

2. The aim of this exercise is to show that $\zeta(s)$ does not vanish on the line $\mathrm{Re}(s) = 1$.

a) Show that for all $\theta \in \mathbb{R}$,

$$3 + 4\cos\theta + \cos 2\theta \geq 0.$$

b) Define

$$D(s) = \zeta(s)^3\zeta(s+it)^4\zeta(s+2it).$$

Show that $D(1) \neq 0$.

c) Conclude that $\zeta(s)$ does not vanish on the line $\mathrm{Re}(s) = 1$.

3. Let

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

Prove that

$$\psi(x) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(- \frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^s}{s} ds.$$

4. Show that

$$\psi(x) = x + \sum_{k=1}^{\infty} \frac{x^{-2k}}{2k} - \log(2\pi) - \lim_{T\to\infty} \sum_{\substack{\varrho \text{ non-trivial} \\ \text{zero of } \zeta \text{ with} \\ |\zeta(\varrho)|<T}} \frac{x^\varrho}{\varrho}.$$

5. Give a prove of the prime number theorem, i.e., show that

$$\psi(x) = x + o(x).$$

Submission: Monday, 30th November 2015 during the exercise class.