# Exercise Sheet 11

**1.** a) Let

$$a_k(n) = \begin{cases} \frac{k!}{\alpha_1!\cdots\alpha_r!} & \text{if } n = p_1^{\alpha_1}\cdots p_r^{\alpha_r} \text{ with } p_i \le X \text{ and } \alpha_1+\ldots+\alpha_r = k, \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\sum_{m\neq n} a_k(m)a_\ell(n) \ll X^{k+\ell}.$$

b) Define

$$\mathcal{P}_0(\sigma_0+it) = \sum_{p\le X} \frac{1}{p^{\sigma_0+it}}.$$

Show that

$$\int_T^{2T} \mathcal{P}_0(\sigma_0+it)^k \mathcal{P}_0(\sigma_0-it)^\ell dt = T\sum_{n\ge 1}\frac{a_k(n)a_\ell(n)}{n^{2\sigma_0}} + O\Big(\sum_{m\neq n}\frac{a_k(m)a_\ell(n)}{(mn)^{\sigma_0}}\frac{1}{|\log(m/n)|}\Big).$$

c) Let $T$ be a large enough number and let

$$\sigma_0 = \frac{1}{2}+\frac{W}{\log T} \qquad W = (\log\log\log T)^4$$
$$X = T^{1/(\log\log\log T)^2} \qquad Y = T^{1/(\log\log T)^2}$$

and suppose that $k$ and $\ell$ are non-negative integers with $X^{k+\ell}\le T$. Show that if $k\neq \ell$, then

$$\int_T^{2T} \mathcal{P}_0(\sigma_0+it)^k\mathcal{P}_0(\sigma_0-it)^\ell \ll T$$

and if $k=\ell$, we have

$$\int_T^{2T} |\mathcal{P}_0(\sigma_0+it)|^{2k} dt = k!T(\log\log T)^k + O_k\big(T(\log\log T)^{k-1}\big).$$

**2.** For non-negative integers $h$ and $k$, we denote by $(h,k)$ the greatest common divisor of $h$ and $k$ and by $[h,k]$ the least common multiple of $h$ and $k$.

**Bitte wenden!**

a) Show that for every squarefree positive integer $n$

$$1 = \sum_{\substack{h,k \\ [h,k]=n}} \mu\big((h,k)\big).$$

b) Let $Y \geq 2$. Show that

$$\sum_{\substack{h,k \\ p|hk \Rightarrow p\leq Y}} \frac{\mu(h)\mu(k)}{(hk)^s}(h,k)^s = \prod_{p\leq Y}\left(1-\frac{1}{p^s}\right).$$

**3.** Let $\mathbb{F}_q$ be a finite field with $q$ elements, $\psi$ an additive character and $\chi$ a multiplicative character of $\mathbb{F}_q$. The Gauss sum associated to $\psi$ and $\chi$ is defined to be

$$\tau(\chi,\psi) = \sum_{x\in\mathbb{F}_q^\times} \chi(x)\psi(x).$$

Show that if $\psi$ and $\chi$ are non-trivial, then

$$|\tau(\chi,\psi)| = \sqrt{q}.$$

**Hint:** Consider $|\tau(\chi,\psi)|^2$ and expand.

**4.** Let $\mathbb{F}_q$ be a finite field with $q$ elements, and let $\chi$, $\varphi$ be multiplicative characters of $\mathbb{F}_q$. The Jacobi sum associated to $\chi$ and $\varphi$ is given by

$$J(\chi,\varphi) = \sum_{x\in\mathbb{F}_q} \chi(x)\varphi(1-x) = \sum_{\substack{x,y\in\mathbb{F}_q \\ x+y=1}} \chi(x)\varphi(y).$$

a) Show that if $\chi$, $\varphi$ and $\chi\varphi$ are all non-trivial, then for any non-trivial additive character $\psi$ of $\mathbb{F}_q$ we have

$$J(\chi,\varphi) = \frac{\tau(\chi,\psi)\tau(\varphi,\psi)}{\tau(\chi\varphi,\psi)}.$$

**Hint:** Consider $J(\chi,\varphi)\tau(\chi\varphi,\psi)$ and expand.

b) Conclude that $|J(\chi,\varphi)| = \sqrt{q}$.

**5.** Let $p$ be a prime number such that $p \equiv 1 \mod 4$. Show that there exist integers $a$, $b$ such that

$$p = a^2 + b^2.$$

**Submission: Monday, 7th December 2015 during the exercise class.**