Exercise Sheet 12

1. a) Let $p \geq 3$ be a prime, $m, n \in \mathbb{Z}$. The associated Kloosterman sum $S(m,n;p)$ is defined by

$$S(m,n;p) = \sum_{x \in \mathbb{F}_p^\times} e\Big(\frac{mx + n\overline{x}}{p}\Big)$$

where $\overline{x}$ is the inverse in $\mathbb{F}_p$ of the invertible element $x \in \mathbb{F}_p^\times$. For $k \geq 0$, let

$$M_k = \frac{1}{(p-1)^2} \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} |S(m,n;p)|^{2k}.$$

Clearly, $M_0 = 1$. Show that for any $k \geq 1$,

$$M_k = \frac{p^2}{(p-1)^2} |\mathcal{A}_k(\mathbb{F}_p)| - \frac{2}{p-1} - \frac{(p-1)^{2k}}{(p-1)^2}$$

where

$$\mathcal{A}_k(\mathbb{F}_p) = \Big\{ (x,y) \in \mathbb{F}_p^\times \times \mathbb{F}_p^\times \Big| \sum_{1 \leq i \leq k} x_i = \sum_{1 \leq i \leq k} y_i \text{ and } \sum_{1 \leq i \leq k} \overline{x_i} = \sum_{1 \leq i \leq k} \overline{y_i} \Big\}.$$

b) Show that

$$|\mathcal{A}_2(\mathbb{F}_p)| = 3(p-2)(p-1).$$

Hence

$$M_2 = \frac{2p^3 - 3p^2 - 3p - 1}{p-1}.$$

c) Show that for all $b \in \mathbb{F}_p^\times$,

$$S(m,n;p) = S(mb, n\bar{b}; p).$$

d) Let m, n be integers coprime with p. Prove the upper bound

$$|S(m,n;p)| \leq 2p^{\frac{3}{4}}.$$

2. a) For $1 \leq k \leq \frac{q}{2}$, show that

$$\Big| q - e\Big(\frac{k}{q}\Big) \Big| \geq c \frac{k}{q},$$

for some absolute constant c.

Bitte wenden!

b) Show that

$$\sum_{k=1}^{q-1} \left| \sum_{n \leq X} e\left(\frac{kn}{q}\right) \right| \ll q \log q.$$

c) Let χ be a non-trivial multiplicative character modulo q. Show that for every $X \geq 0$,

$$\sum_{n \leq X} \chi(n) \ll \sqrt{q} \log q.$$

3. Let

$$\mu = \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}} dx$$

be the Sato-Tate measure on $[-2, 2]$. Show that the moments satisfy

$$\int_{-2}^{2} x^k d\mu = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \frac{1}{1+\frac{k}{2}} \binom{k}{\frac{k}{2}} & \text{if } k \text{ is even.} \end{cases}$$

No submission.