Exercise Sheet 2

1. Give a proof of the fundamental theorem of arithmetic: Every integer $n \geq 1$ can be represented as a product of prime factors in only one way, apart from the order of the factors.

Hint: You can use induction over n, the case $n = 2$ being trivial.

2. Let f be an arithmetic function and assume that the sum $\sum_{n\geq 1} f(n)n^{-s}$ converges absolutely for $\sigma > \sigma_0$. Show that

a) if f is multiplicative, then

$$\sum_{n\geq 1} \frac{f(n)}{n^s} = \prod_p \left(\sum_{k\geq 0} \frac{f(p^k)}{p^{ks}} \right)$$

for $\sigma > \sigma_0$.

b) if f is completely multiplicative, then

$$\sum_{n\geq 1} \frac{f(n)}{n^s} = \prod_p \frac{1}{1 - f(p)p^{-s}}$$

for $\sigma > \sigma_0$.

Note: This identities are called Euler products.

3. Show that

$$\sum_{p\leq x} \frac{1}{p} = \log\log x + O(1).$$

Hint: Feel free to use the prime number theorem in your proof.

4. a) Show that the “probability” that an integer n is squarefree is $\frac{6}{\pi^2}$, i.e., show that

$$\lim_{N\to\infty} \frac{1}{N} |\{1 \leq n \leq N \mid n \text{ is squarefree}\}| = \frac{6}{\pi^2}.$$

b) Compute the “probability” that two integers n_1 and n_2 are coprime, i.e., compute

$$\lim_{N\to\infty} \frac{1}{N^2} |\{(n_1, n_2) \in \Omega_N \times \Omega_N \mid (n_1, n_2) = 1\}|$$

where $\Omega_N = \{1, \ldots, N\}$.

Submission: Monday, 5th October 2015 during the exercise class.