Prof. Dr. Emmanuel Kowalski

Exercise Sheet 4

1. Let $P \in \mathbb{Z}[X]$ be a polynomial of degree m. Consider $k \in \mathbb{Z}$ with $k > m$. Show that

$$\frac{1}{N}|\{1 \leq n \leq N \mid P(n) \text{ is } k\text{-free}\}| = \sum_{d \leq cN^{\frac{m}{k}}} \mu(d)\nu_{d^k}(0) + O_P(N^{\frac{m}{k}-1})$$

for some c depending only on P, where $\nu_{d^k}(0)$ is given by

$$\nu_{d^k}(0) := \frac{1}{d^k}|\{x \in \mathbb{Z}/d^k\mathbb{Z} \mid P(x) \equiv 0 \mod d^k\}|.$$

2. Let $P \in \mathbb{Z}[X]$ be a polynomial of degree m and let $k \geq 2$.

a) Show that

$$\mu(d)\nu_{d^k}(0) \leq \frac{m^{\omega(d)}}{d^k}(d, \Delta)^{k-1},$$

where Δ denotes the discriminant of P.

b) Show that

$$\sum_{d=1}^{\infty} \mu(d)\nu_{d^k}(0)$$

converges absolutely. Conclude that

$$\sum_{d=1}^{\infty} \mu(d)\nu_{d^k}(0) = \prod_p \big(1 - \nu_{p^k}(0)\big)$$

3. For $P(X) = X^2 + 1$, show that

$$\nu_{p^3}(0) := \frac{1}{p^3}|\{x \in \mathbb{Z}/p^3\mathbb{Z} \mid P(x) \equiv 0 \mod p^3\}| = \begin{cases} \frac{2}{p^3} & \text{if } p \equiv 1 \mod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Compute

$$\lim_{N\to\infty} \frac{1}{N}|\{1 \leq n \leq N \mid n^2 + 1 \text{ is 3-free}\}|.$$

Submission: Monday, 19th October 2015 during the exercise class.