Exercise Sheet 5

1. Let $(a_n)_{n\geq 1}$ be a sequence of complex coefficients and let $\sigma_0 \in \mathbb{R}$ be such that

$$\sum_{n\geq 1} \frac{a_n}{n^{\sigma_0}}$$

converges. Show that

$$\sum_{n\geq 1} \frac{a_n}{n^{s}}$$

converges uniformly on compact sets in Re$(s) > \sigma_0$.

2. Let $(a_n)_{n\geq 1}$ be a sequence of complex coefficients. If there exists a bound B such that for all $M, N > 0$,

$$\Big| \sum_{n=M}^{M+N} a_n \Big| \leq B.$$

Prove that

$$\sum_{n\geq 1} \frac{a_n}{n^{s}}$$

converges for Re$(s) > 0$.

3. Let $D \subset \mathbb{C}$ be a disc and let (f_n) be a sequence of functions, each holomorphic on D, and let $f_n(z) \to f(z)$ uniformly in D. Prove that if $f_n(z) \neq 0$ for all $n \in \mathbb{N}$ and for all $z \in D$, then either f is identically zero on D or $f(z) \neq 0$ for all z in the interior of D.

4. Assume the following theorem:
Theorem. *Let $m \geq 1$ be an integer. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. Let $(\mathbf{X}_n)_{n\geq 1}$ be $\mathbb{R}^m$-valued random vectors on Ω such that all moments $M_{\mathbf{k}}(\mathbf{X})$ exist, and such that there exist constants $c_{\mathbf{k}} \geq 0$ with*

$$\mathbb{E}(|X_{n,1}|^{k_1} \cdots |X_{n,m}|^{k_m}) \leq c_{\mathbf{k}}$$

for all $n \geq 1$. Assume that $\mathbf{X}_n$ converges in law to a random vector $\mathbf{Y}$. Then $\mathbf{Y}$ is mild and for any m-tuple $\mathbf{k}$ of non-negative integers, we have

$$\mathbb{E}(|X_{n,1}|^{k_1} \cdots |X_{n,m}|^{k_m}) \to \mathbb{E}(|Y_1|^{k_1} \cdots |Y_n|^{k_m}).$$

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $(X_n)_{n\geq 1}$ be $\mathbb{R}$-valued random variables on Ω given by

$$X_n = \frac{B_1 + \ldots + B_n}{\sigma_n}$$

where the variables $(B_n)_{n\geq 1}$ are independent and satisfy

$$\mathbb{E}(B_n) = 0, \quad |B_n| \leq 1, \quad \sigma_n^2 = \sum_{i=1}^{n} \mathbb{V}(B_n) \to +\infty.$$

Assume that X_n converges in law to a random variable Y. Show that Y is mild (using the theorem).

Submission: Monday, 26th October 2015 during the exercise class.